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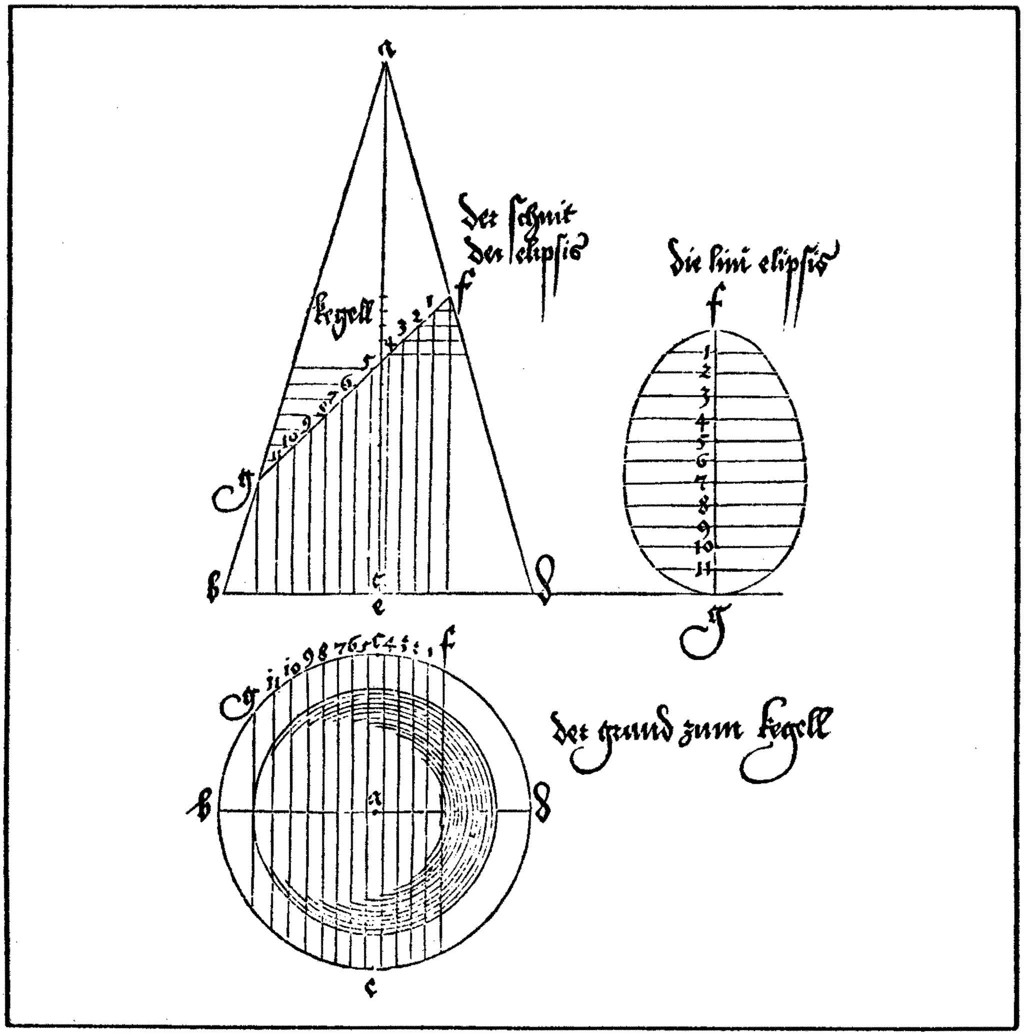
MATHEMATICS MAGAZINE



- Dürer's Paradox
- The Census-Taker Problem
- Walks Guided by the Sun

An Official Publication of The MATHEMATICAL ASSOCIATION OF AMERICA

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The full statement of editorial policy appears in this *Magazine*, Vol. 54, pp. 44–45, and is available from the Editor. Manuscripts to be submitted should not be concurrently submitted to, accepted for publication by, nor published by another journal or publisher.

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AUTHORS

Roger Herz-Fischler started life as a theoretical probabilist but switched to another random path after teaching a course for architecture students. The origin of his article on Dürer was the skepticism of a student in that course. That course also led to a recent book on the mathematical history of the "golden number" and current research on the history, sociology and veracity of its quasi-mathematical aspects.

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ARTICLES

Dürer's Paradox or Why an Ellipse Is Not Egg-Shaped

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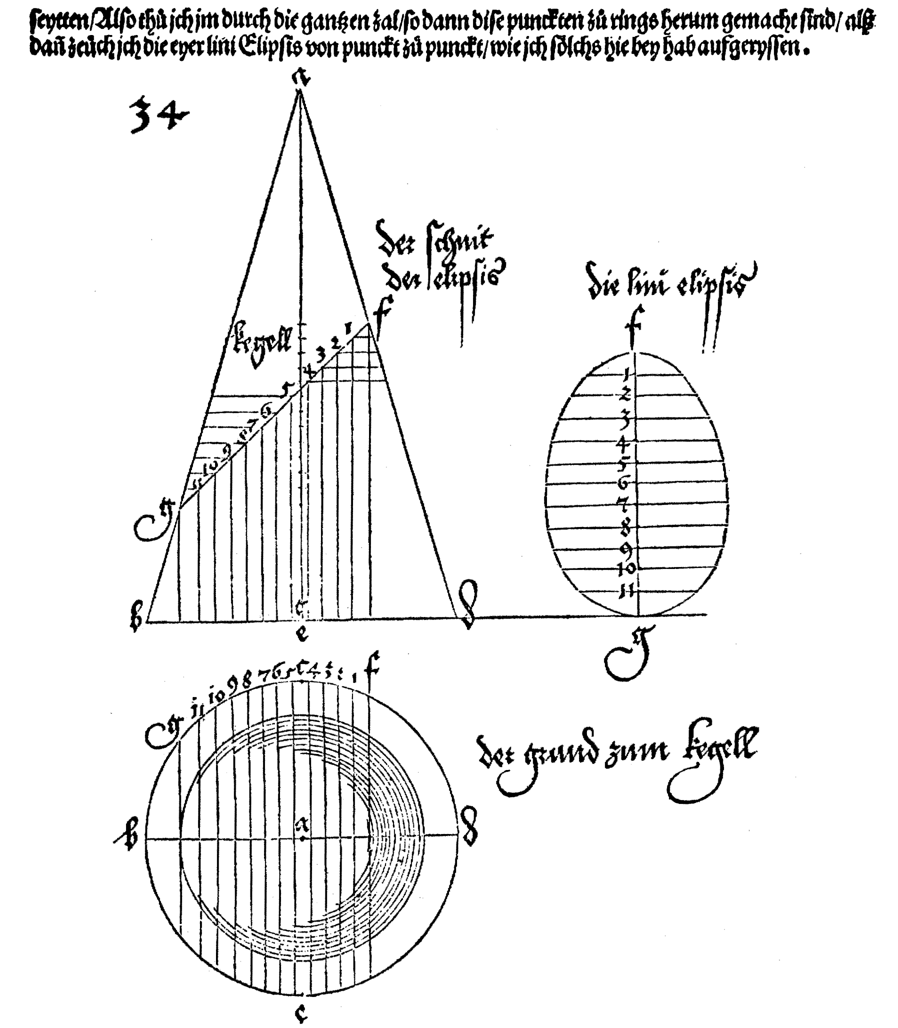
"The ellipse I will call an egg-curve because it is virtually equal to an egg"
—A. Dürer

The drawing of the cone and the ellipse is taken from the 1525 work *Treatise on Mensuration with the Compass and Ruler in Lines, Planes, and Whole Bodies* by the German artist and mathematician Albrecht Dürer (1471–1528). Two things are likely to attract the attention of the viewer: namely, the mass of lines and arcs and—to use Dürer's own expression—the "egg-curve." The purpose of this article is threefold: to describe Dürer's method, to explain why the use of the method might lead one to believe that the ellipse is egg-shaped, and to show how the analytic version of Dürer's method can be used to derive the Cartesian form of the ellipse. I have also included some historical material and suggestions for further reading in a separate section at the end of the article.

The method used by Dürer is essentially equivalent to what is now called descriptive geometry. At present it is only employed to obtain graphical solutions of complicated geometrical problems such as the intersection of surfaces, and as far as I know this branch of mathematics is now taught only in engineering drawing courses. However the French geometer Gaspard Monge (1746–1818), whose *Géométrie descriptive* provided the systematic development of the subject, wrote [**11**, p. 1]: "[One of the aims of descriptive geometry] is to give means of recognizing, based on an exact description, the forms of bodies and to deduce all the truths which are implied by their form and their respective positions."

Descriptive geometry is concerned with the representation of bodies and surfaces in space by means of two-dimensional orthogonal projections. Consider for example FIGURE 1 in which a point P is shown as being a units in front of a vertical plane and b units above a horizontal plane. If we project the point P onto these planes by means of perpendicular lines then the two projections P^V and P^H are both determined. To avoid constant awkward repetition and notation, a point and its projections will be referred to by a single letter without super and subscripts. If the horizontal plane is now folded back about the "folding line"— the line of intersection of the two planes —until the horizontal plane is also in a vertical plane, then we will have the situation of FIGURE 2. Here the point P is represented by two two-dimensional drawings which are such that the two projections lie on a line perpendicular to the folding line. Furthermore given the two-dimensional drawings, which are usually referred to as the front and top views, then the position in space—relative to the two planes—of the point P is completely determined. The folding line is not really necessary and if we are only interested in the relative position of various points with respect to one



another then any convenient line can be used. This is the case in Dürer's drawing where the centre line of the circle is used as the reference line in the top view.

In the simple case just discussed we are given the distance of the point P from the planes, but in general life is not so simple. Sometimes, for example when one is dealing with a complicated intersection of two three-dimensional objects, the process of obtaining the two views is quite involved. Fortunately, in the case that interests us we are only dealing with a plane and a cone.

Thus suppose (FIGURE 3) that we have a cone that has been cut by a horizontal plane. Then in the front view the cone will appear as a triangle and the intersection of the cutting plane with the cone will appear as a straight line. On the other hand in the top view the intersection of the cutting plane and the cone will be a circle of some radius r. Let O be the centre of the circle and consider the horizontal centre line MON. Since the points M and N are at the extreme right and left respectively of the circle the same will be true of the front view of the centre line. Thus the line in the front view can also be thought of as representing the centre line of the circle, and so

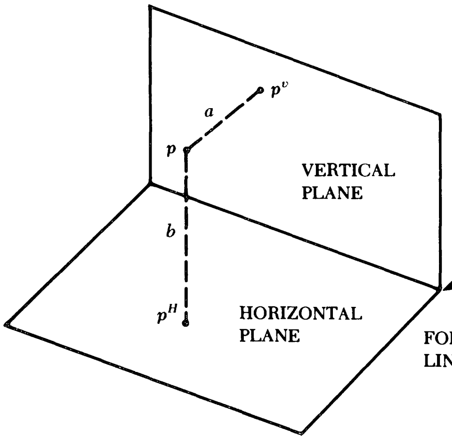


FIGURE 1

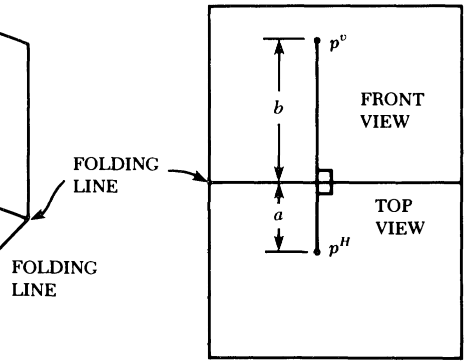


FIGURE 2

we also label this line as MN. Because the two projections of M lie above one another, they will both be at the same distance r from the centreline of the cone. This in turn means that in order to draw the circle in the top view we need only measure the distance r in the front view and then use this as the radius of a circle, about O, in the top view. If we have an arbitrary point P on the centre line MON then this will determine the points P' and P'' on the circumference and all three points will coincide in the front view.

Now let us cut the cone with an oblique plane (FIGURE 4). The intersection of the plane with the cone determines a curve which, because of previous knowledge, will be referred to as the ellipse. The front view of the plane is once again a straight line and by the symmetry of the cone this line FG also corresponds to the major axis of the ellipse. Because the cutting plane is at an angle with the horizontal the ellipse will appear to be distorted when looked at from directly above; it is only when we look at right angles to the cutting plane (FIGURE 4) that the ellipse appears in its true shape. It is for this reason that the top and front views must be used to obtain a new (auxiliary) view, such as the one that appears in Dürer's drawing, which shows the true shape of the ellipse.

In order to find points belonging to the top view of the ellipse we pass a horizontal cutting plane through the cone which intersects the given oblique plane at an arbitrary height (FIGURE 4). This brings us back to the situation of FIGURE 3 and given the radius—or equivalently the location of a point on the major axis FG—the circle determined by the horizontal plane can be drawn. Once again we designate the centre line of this circle determined by the horizontal plane as MN. But this circle is located on the surface of the cone, as is the ellipse, and so the circle and ellipse intersect in two points P' and P''. In the front view we will see these two points of intersection of the ellipse and the cone as the intersection of lines FG and MN. This observation in turn determines the location of P' and P'' in the top view, for these points must lie on both the circle and the vertical line drawn down from P', P'' in the front view. In particular the width ($2w$) of the ellipse at the point P on the major axis of the ellipse which corresponds to the points P' and P'' is now determined.

By repeating the above process we can find the width of the ellipse for as many points on the major axis as we wish. This process is illustrated in FIGURE 5 for two points 1 and 2 which are equidistant from the centre O of the centreline FG. In Dürer's drawing the major axis FG of the ellipse has been divided into twelve equal

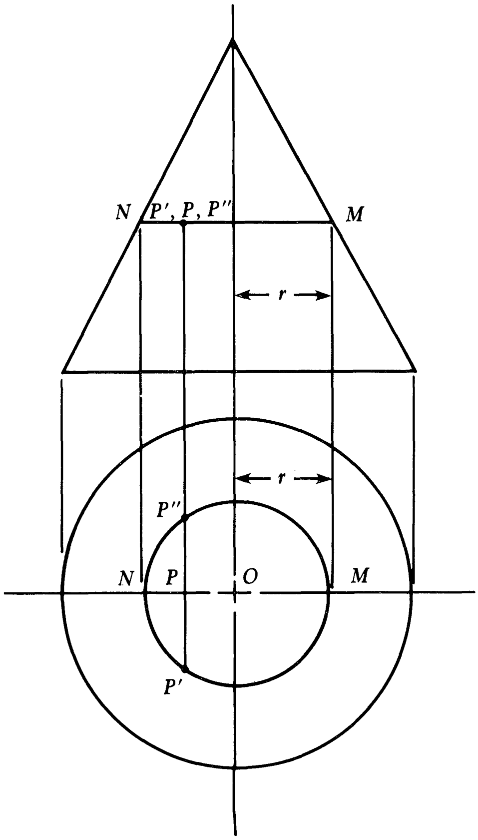


FIGURE 3

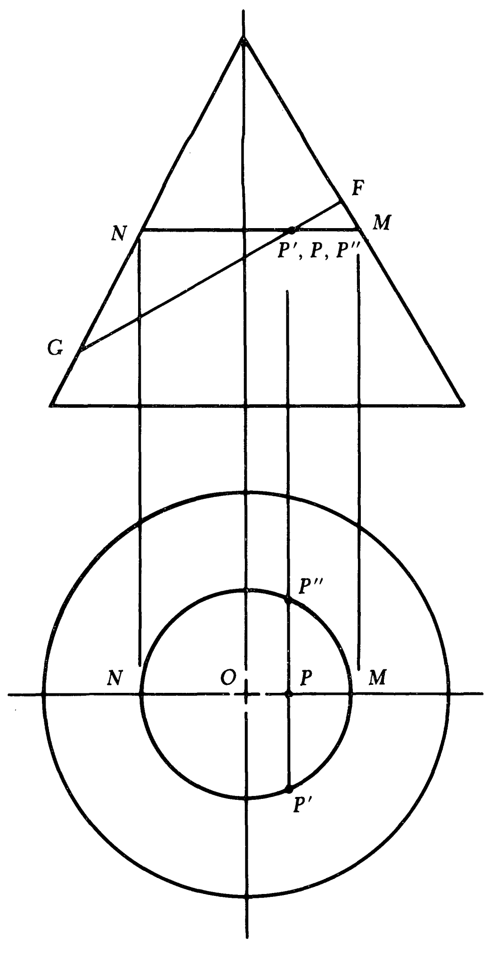


FIGURE 4

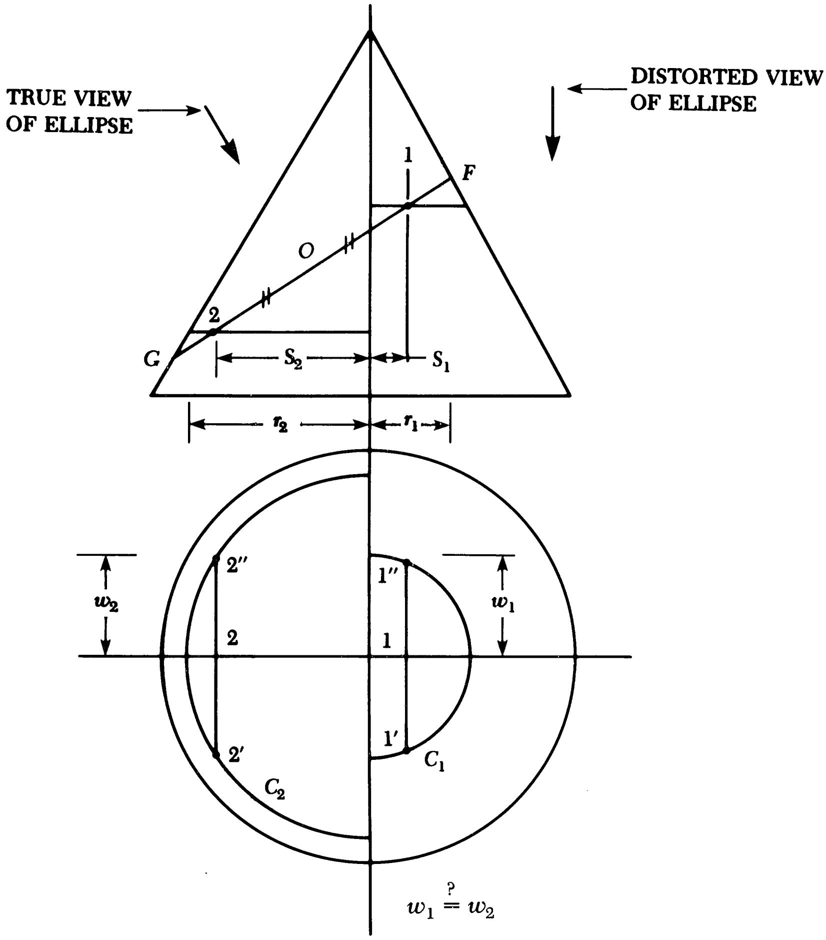


FIGURE 5

parts by means of the points labeled 1, . . . , 11. A horizontal line drawn through each of these points in the front view corresponds to a horizontal slicing circle and these circles are then drawn in the top view as shown in Dürer's diagram. The downward projection of each of the eleven points on the corresponding circle now gives the width of the ellipse at that point.

Note now that even though points 1 and 11 are symmetrically located with respect to the centreline the corresponding constructions are not symmetric, for point 1 corresponds to the smallest of the eleven circles whereas point 11 corresponds to the largest. Thus it is not at all obvious from this approach that the ellipse is symmetric rather than egg-shaped, with the wider part corresponding to point 11. If one did not know the analytic form of an ellipse then it might seem more reasonable, using Dürer's approach to conclude that the ellipse is indeed egg-shaped. This not unreasonable conclusion is what I refer to in the title as Dürer's paradox.

Despite the apparent paradox we do know that the ellipse is *not* egg-shaped, and the key to understanding why is the fact that the centreline of the cone does not pass through the centre of the ellipse. With reference to FIGURE 5 let O be the centre of the line FG and let 1 and 2 be two points on FOG which are equidistant from O. Let r_1 and r_2 be the radii of the corresponding circles C_1 and C_2 and s_1 and s_2 the horizontal distances from the centreline of the cone. It is true that $r_1 < r_2$, but because 1 and 2 are equidistant from O we also have that $s_1 < s_2$, i.e., the point 1 is closer to the centre of the smaller circle C_1 than point 2 is to the centre of the larger circle C_2. The reason the ellipse is not egg-shaped is that the combination of shorter distance to the centre and smaller radius combine to make $w_1 = w_2$. Of course I—Dürer does not go through any reasoning of this nature—have flippantly stated that $w_1 = w_2$ because I know that this is the way things work out for an ellipse. What is needed is a proof of this fact.

Thus what I will now do is translate Dürer's construction, which involves a finite number of points, into an analytic argument and obtain expressions for w_1 and w_2 when points 1 and 2 are symmetrically located with respect to the centre O. It will then be shown that not only are the widths equal, but also that we can obtain the usual Cartesian equation of the ellipse from the expression for the width. Furthermore the relationships between the constants and the angles of the cone and cutting plane, as well as some other results, will fall out of the derivation. While the development is in principle straightforward, a nonjudicious choice of parameters can lead to some very messy algebra. After some trials the following seems to me to be the simplest approach.

We start with a cone whose cone angle is θ and then pass a cutting plane which makes an angle $\alpha > \theta$ with the centreline of the cone. Let FG be the major axis of the ellipse with centre O and length $2a$. We take arbitrary points 1 and 2 which are at a distance x from O and designate by s and t the horizontal and vertical displacements from O (FIGURE 6a). Thus:

$$s = x \sin\alpha; \qquad t = x \cos\alpha. \tag{1}$$

If d^* is the horizontal displacement of O from the centreline, then the points 1 and 2 are respectively at a horizontal distance $s - d^*$ and $s + d^*$ from the centreline. If r^*, r_1, r_2 are the radii corresponding to points O, 1, 2, then as indicated at the bottom of FIGURE 6a we have the relationships:

$$r_1 = r^* - t\cdot\tan\theta; \qquad r_2 = r^* + t\cdot\tan\theta. \tag{2}$$

Now looking at the top view (FIGURE 6b) and considering the circles of radii r_1 and r_2, we see that w_1 and w_2—the respective half-widths of the ellipse corresponding to points 1 and 2—satisfy:

$$w_1^2 = r_1^2 - (s - d^*)^2 = (r^* - t\cdot\tan\theta)^2 - (s - d^*)^2 \tag{3}$$

$$w_2^2 = r_2^2 - (s + d^*)^2 = (r^* + t\cdot\tan\theta)^2 - (s + d^*)^2. \tag{4}$$

What we need in order to continue are expressions involving r^* and d^*. These may be obtained by considering the triangles OFH and OJG (FIGURE 7) and then applying the law of sines to each. This gives the two relationships:

$$r^* + d^* = (a/\cos\theta)\cdot\sin(\alpha + \theta) \tag{5}$$

$$r^* - d^* = (a/\cos\theta)\cdot\sin(\alpha - \theta). \tag{6}$$

Adding and subtracting and applying trigonometric reduction formulae we obtain after simplification:

$$r^* = a\cdot\sin\alpha \tag{7}$$

$$d^* = a\cdot\cos\alpha\cdot\tan\theta. \tag{8}$$

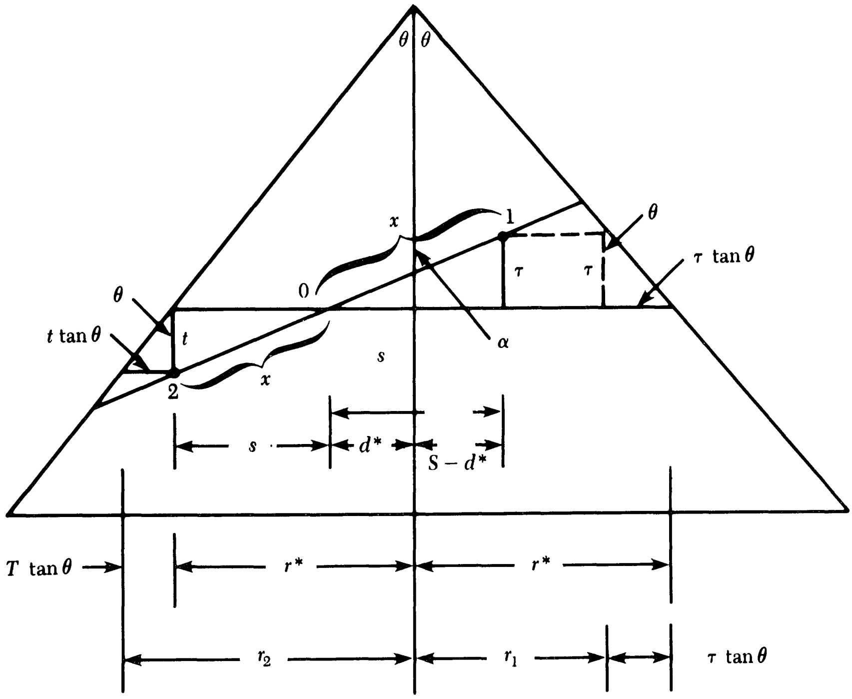


FIGURE 6a

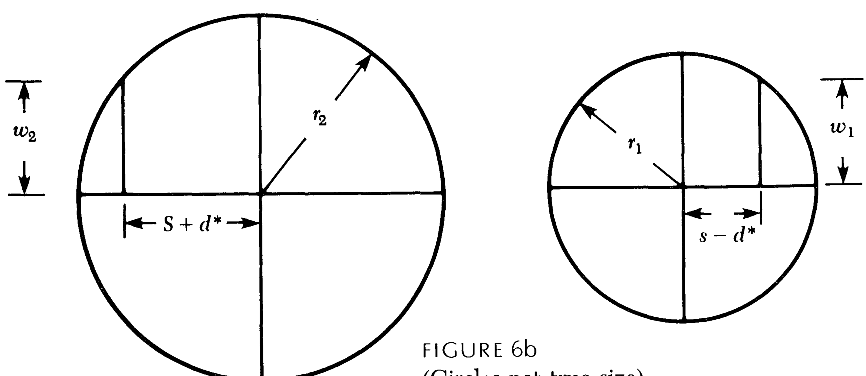


FIGURE 6b
(Circles not true size)

FIGURE 6

To obtain the equation of the ellipse—recall that we know by symmetry of the cone that the ellipse is symmetric about the major axis—we substitute (7) and (8) in (3) and (4) to obtain:

$$w_1^2 = (a \cdot \sin\alpha - x \cdot \cos\alpha \cdot \tan\theta)^2 - (x \cdot \sin\alpha - a \cdot \cos\alpha \cdot \tan\theta)^2 \qquad (9)$$

$$w_2^2 = (a \cdot \sin\alpha + x \cdot \cos\alpha \cdot \tan\theta)^2 - (x \cdot \sin\alpha + a \cdot \cos\alpha \cdot \tan\theta)^2. \qquad (10)$$

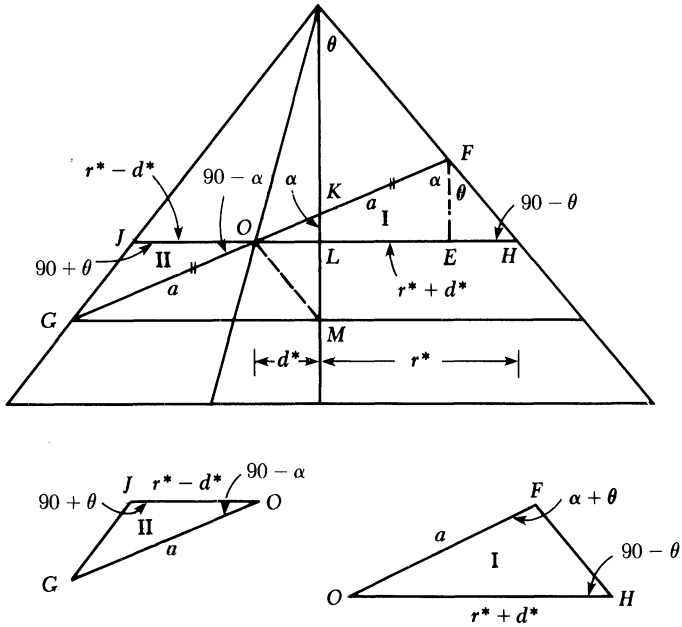


FIGURE 7

Since the expressions in (9) and (10) only differ by the signs inside the parentheses and since the inner terms in the expansions of the squared terms will cancel in each case, we have immediately—without any calculations—that $w_1 = w_2$, i.e., that the ellipse is indeed symmetric.

If we designate the common width at a distance x along the major axis by y and simplify we obtain:

$$y^2 = (a^2 - x^2)[\sin^2\alpha - \cos^2\alpha \cdot \tan^2\theta] \text{ or} \tag{11}$$

$$y^2/b^2 + x^2/a^2 = 1 \text{ where} \tag{12}$$

$$b^2 = a^2[\sin^2\alpha - \cos^2\alpha \cdot \tan^2\theta] = a^2\left[1 - (\cos\alpha/\cos\theta)^2\right]. \tag{13}$$

If the eccentricity is defined by $e = \cos\alpha/\cos\theta$, then (13) is just the usual relationship between the semi-axes of an ellipse so that (12) does indeed agree with the standard Cartesian equation of the ellipse. Here the eccentricity can be thought of directly as a measure of the distortion of the circle (corresponding to $\alpha = 90°$) due to the tilting of the slicing plane.

There is some more information that we can easily obtain from the above development. As an immediate consequence of (7) and (8) we have that if we draw the perpendicular FE in triangle OFH (FIGURE 7) then:

$$OE = r^*; \qquad EH = d^* \tag{14}$$

and further:

$$r^*/d^* = \tan\alpha/\tan\theta. \tag{15}$$

I was unable to obtain either (14) or (15) by more direct trigonometric or geometric means.

If K is the intersection of the centre line of the cone with the axis FG then by similar triangles we obtain:

$$OK/a = OL/OE = d^*/r^* = \tan\theta/\tan\alpha. \qquad (16)$$

This latter relationship incidentally implies that the right focal point is to the right of point K, i.e., the centreline of the cone passes strictly between the centre of the ellipse and the "upper" focal point, for if $c = ae = a(\cos\alpha/\cos\theta)$ is the focal distance, then (16) implies that $c/OK = \sin\alpha/\sin\theta > 1$. This relationship between c and OK can also be obtained by drawing the line OM—whose length turns out to be equal to c—and then applying the law of sines to both triangles OMG and OKM.

Finally note that Dürer also constructed the parabola ("burn-curve") and the hyperbola ("fork curve"; the drawing is for the special case where the cutting plane is vertical) and the reader is invited to make the drawings—before looking at [5]—and to obtain the Cartesian equations for these conic sections.

Historical Notes and Further Reading

The diagram was taken from the first edition of [5] which has been reprinted twice in recent years, once with English translation and commentary. Just before the drawing (number 34 of Book I; this is on page 94 of the English edition) Dürer explains how the construction proceeds, but unfortunately he gives no historical information as to the origin of the method. He merely states at the beginning that the ancients, i.e., the Greeks, showed that three different curves are obtained when a cone is cut by different planes. Dürer then informs the reader that the learned names are "Elipsis," "Parabola" and "Hiperbole," but that he does not know the German names. He says, "We want to give them names which in themselves will serve for identification purposes." As Dürer states in the quotation given at the beginning of the article he calls the ellipse an egg-curve [eyer lini = eierlinie] because the ellipse is virtually [schyer = schier] equal to an egg.

As indicated by Dürer's remark about "the ancients" he was acquainted with the work of the Greek geometers. That he was well versed in the geometry of Euclid and others is evident from the various constructions throughout the book. There also exists other evidence relating to Dürer's mathematical studies and knowledge. The mathematical facet of Dürer's life is generally not known, even to people who are acquainted with his engraving "Melancolia" (reproduced in Boyer [**1**, p. 325]) which shows a magic square and a polyhedron. While a rarity in our day, many Renaissance artists had an advanced knowledge of mathematics. Another example of a great artist with a knowledge of mathematics is the Italian artist Piero de la Francesca (c. 1415 to 1492) who wrote several interesting treatises on perspective and mathematics (see [**9**, Section 31, C]). Staigmüller [**16**, p. 3] wrote that the lack of knowledge of Dürer's mathematical work among art historians was surprising in view of the emphasis that Dürer himself put on it and that when he had laid down his brush he wrote his theoretical works in order to pass on his knowledge. "Indeed he lived in the hope that through these works, even more than through his eternal creations with the brush, he would lay the foundation of the 'German art' because he regarded the lack of a theoretical, particularly mathematical, knowledge in his fellow artists as the main hinderance to a prosperous development of the arts in the fatherland."

On the life and work of Dürer, see [**17**, p. 35]; [**18**]; [**19**]; [**10**]; [**3**, p. 109]; [**4**, p. 61]; [**20**, p. 62]. The art of Dürer is discussed by Panofsky [**12**] who devotes his chapter 8 to Dürer as a theorist of art.

Of particular historical interest to us are Dürer's method of construction and the shape of his ellipse and I will briefly indicate what is known.

The history of the study of the conic sections in the Middle Ages and the Renaissance is examined in great detail by Clagett [2]. He suspects [p. 266] that Dürer was influenced by the 15th-century mathematician Regiomantus, but not by Dürer's contemporary Johann Werner who had published a book on the conics in 1522. Regarding Dürer's technique Clagett writes: "The method...has no counterpart in the medieval traditions of conic sections, nor indeed in the revived Apollonian tradition that was soon to follow."

The egg-shaped ellipse has been commented upon by several writers. Staigmüller [**16**, p. 16] believed, based on the drawing and the name egg-curve that he gave it, that Dürer thought that the ellipse only had one axis of symmetry and that it was wider at the end that was at the bottom of the cone. Steck [**17**, p. 35] also commented on this question and gave references to other authors (Doehlmann and Günther), but none of these has any precise information to give on this question. Hofmann [**10**, p. 119] mentions other authors who have an egg-shaped ellipse, but these are all in works published after Dürer except for Witelo (13th century; see [**2**, Chapter 3]). I checked [**20**], but did not find any sign of an egg-shaped ellipse. Typically Book 7, proposition 47 shows a cone and a crudely drawn ellipse in perspective. Coolidge [**3**] merely states "It is fair to state that Dürer's ellipse looks rather egg-shaped."

While reading Pottage's [**15**] treasure chest of geometrical and historical information my attention was drawn to Pedoe [**14**]. Pedoe seems to suggest that the egg shape can be accounted for on the basis of errors in the method of reproducing the drawing. He points out that, in terms of FIGURE 3 of this article, that $(PP'')^2=(PN)(PM)$ (because PP'' is the altitude of right triangle $NP''M$). He then says, without giving any further details, that the equations of the conic sections follow from Dürer's method if one uses a non-specified theorem on similar triangles. This method, says Pedoe, is the method of Apollonius [see **22**, p. 288 ff.]. Thus in view of Dürer's acquaintance with the ancients, Pedoe suggests that Dürer extracted his practical method for constructing the conic sections from the work of Apollonius.

The above survey suggests that we essentially do not know anything precise about the origin of Dürer's technique. It also seems to me that we must presume that Dürer thought that the ellipse was indeed egg-shaped. Whether this was for theoretical reasons or because of his drawing one cannot say. As regards the latter possibility I invite the reader to apply straightedge and dividers to Dürer's drawing and to note certain inaccuracies, particularly in the vertical projections. This seems very strange in the work of one of the master engravers of all time.

For descriptions and discussions of Monge's work see Taton [**20**, **21**]. Monge is responsible for systematizing and advancing earlier work involving graphical techniques and particularly in putting them on a sound mathematical basis. Taton [**21**] writes: "Monge viewed descriptive geometry as a powerful tool for discovery and demonstration in various branches of pure and infinitesimal geometry. His persuasive example rehabilitated the study and use of pure geometry, which had been partially abandoned because of the success of Cartesian geometry."

I have recently advocated ([7], [8]) a computer oriented, algorithmic approach to the teaching of certain types of three-dimensional shape, form and space problems, which is based in part on the descriptive geometry approach—although with the aim of obtaining a numerical answer and not a drawing—and which in a sense follows the spirit advocated by Monge. The methods of descriptive geometry—more or less explained—and in particular the construction of the ellipse can be found in various "technical drawing" or "descriptive geometry" books. See for example [**13**, chapter 21] and [**6**, section 337].

Acknowledgements. First of all, I must thank the anonymous student in my Mathematics for Architecture class—see [7]—who shook me out of my complacency by asking out loud, at the moment when I presented Dürer's drawing, why indeed he should not believe that the ellipse was egg-shaped. This forced me to think about the matter and this in turn led to the present analytic argument. I do not recall now if the student was convinced! Secondly, I wish to express my appreciation to one of the referees, who obviously devoted a great deal of time to my manuscript, for some very pertinent remarks and corrections and for providing me with the reference to the Strauss edition of Dürer. I would also like to thank Dr. Vera Huse now of Acadia University for help with the intricacies of the German language and to Dr. Gert Schubring of the Institut für Didaktik der Mathematik, Universität Bielefeld, for obtaining a copy of Staigmüller's 1891 "Schulprogramm" for me.

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NOTES

The Census-Taker Problem

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A census taker comes to a two-family house. After obtaining the required information from a downstairs resident, the census taker asks, "Does anyone live upstairs?"

"Yes, but everyone is out now. However, I can supply you with the necessary information."

"How many people live upstairs?"

"Three."

"What are their ages, to last birthday?"

"Well," the downstairs resident answers in puzzle-ese, "the product of their ages is 1296, and the sum of their ages is the house number, which you already know."

The census taker, who never wastes questions, computes for a while and then asks, "Is anyone living upstairs older than you?"

"Yes."

"Thank you. I now know the ages."

What are the ages?

The initial reaction of someone encountering the problem for the first time is likely to be that insufficient information is given. How can one determine which of the 42 triples with product 1296 (see TABLE 1) is the correct one? What does the unspecified house number have to do with the problem?

According to Greenblatt [2], the problem apparently originated during World War II, and it has appeared in print several times since then (see [1–4]). Several other problems with seemingly insufficient information have been proposed, such as [3, 5–7]. The first author heard this problem in 1951.

The key to the problem is to put oneself in the census taker's place. If the house number had been any number in the "Sum" column of TABLE 1 *except* 91, then there would have been no need for the final, apparently irrelevant, question, since the census taker could have determined the ages from knowledge of the house number alone. Hence, the house number must be 91, and the extra question is required to distinguish between the triples $(81, 8, 2)$ and $(72, 18, 1)$. Since the census taker knows that the age of the downstairs resident is between 72 and 81 (but we don't!), the answer to the extra question determines that $(81, 8, 2)$ is the correct triple.

The number 1296 has appeared in almost all published statements of the problem. In 1961, we decided to investigate the possibility of using other numbers, and first made a hand calculation by trying out consecutive integers one by one. We gradually noticed some characteristics shared by the "census-taker numbers," and these characteristics developed into the conditions stated below. These conditions were used to

TABLE 1. Age triples for 1296 and their sums.

Ages	Sum	Ages	Sum	Ages	Sum
1296 + 1 + 1 =	1298	81 + 16 + 1 =	98	36 + 12 + 3 =	51
648 + 2 + 1 =	651	81 + 8 + 2 =	91	36 + 9 + 4 =	49
432 + 3 + 1 =	436	81 + 4 + 4 =	89	36 + 6 + 6 =	48
324 + 4 + 1 =	329	72 + 18 + 1 =	91	27 + 24 + 2 =	53
324 + 2 + 2 =	328	72 + 9 + 2 =	83	27 + 16 + 3 =	46
216 + 6 + 1 =	223	72 + 6 + 3 =	81	27 + 12 + 4 =	43
216 + 3 + 2 =	221	54 + 24 + 1 =	79	27 + 8 + 6 =	41
162 + 8 + 1 =	171	54 + 12 + 2 =	68	24 + 18 + 3 =	45
162 + 4 + 2 =	168	54 + 8 + 3 =	65	24 + 9 + 6 =	39
144 + 9 + 1 =	154	54 + 6 + 4 =	64	18 + 18 + 4 =	40
144 + 3 + 3 =	150	48 + 27 + 1 =	76	18 + 12 + 6 =	36
108 + 12 + 1 =	121	48 + 9 + 3 =	60	18 + 9 + 8 =	35
108 + 6 + 2 =	116	36 + 36 + 1 =	73	16 + 9 + 9 =	34
108 + 4 + 3 =	115	36 + 18 + 2 =	56	12 + 12 + 9 =	33

shorten the work when the problem was programmed for computer (IBM 709 and 7094 in 1962 and 1965, IBM 1130 in 1971 and 1972, and Apple Macintosh in 1987), which yielded all census-taker numbers up to 10000.

But first we need a definition. *A census-taker number* is a positive integer N for which there is *exactly one pair* of distinct triples with product N and the same (unspecified) sum. In the following discussion, it will be assumed that N is a census-taker number with distinct triples (A, B, C) and (D, E, F), so that

$$ABC = DEF = N \quad \text{and} \quad A + B + C = D + E + F. \tag{1}$$

Since the order of the triples and the order of the numbers in each triple are irrelevant, we may assume that

$$A \geqslant B \geqslant C, \qquad D \geqslant E \geqslant F, \quad \text{and} \quad C \geqslant F. \tag{2}$$

Using these inequalities and divisibility arguments, we can show successively:

1. The two triples cannot have a common entry.
2. $A > D$, $B < E$, and $C > F$.
3. N is not a prime power.
4. N is the product of at least four primes (not necessarily distinct).

For N having only a small number of divisors, these lemmas enable us to show by examination of cases the form a census-taker number must take. In fact, we have shown that if a census-taker number N has fewer than 12 divisors, then the form of N can be given precisely, as shown in TABLE 2 (p and q being distinct primes). Note

TABLE 2. Census-taker numbers with fewer than 12 divisors.

Number of divisors	Form of N	Condition	Example
8	p^3q	$q = p^2 + p - 1$	$p = 2,\ q = 5,\ N = 40$
9	p^2q^2	$q = 2p - 1$	$p = 2,\ q = 3,\ N = 36$
10	p^4q	$q = p^3 + p^2 - 1$	$p = 2,\ q = 11,\ N = 176$

that if N has exactly 11 divisors or fewer than 8, then it cannot be a census-taker number. (Complete proofs can be obtained by writing to the first author.)

Although necessary conditions that a number with 12 or more divisors be a census-taker number can be found, these conditions are considerably more complicated and take one out of the realm of recreational mathematics. TABLE 3 lists the census-taker numbers through 1296, together with the corresponding house numbers. Related material, generally without the requirement that there be exactly one pair of triples with the same sum, can be found in [9–12].

TABLE 3. Census-taker numbers from 1 through 1296.

N	sum	N	sum	N	sum	N	sum	N	sum
36	13	252	34	600	39	816	59	1092	98
40	14	280	35	648	38	850	44	1160	70
72	14	297	39	690	54	855	65	1188	77
96	21	320	28	714	60	896	73	1216	84
126	26	408	42	735	57	945	39	1242	78
176	28	520	35	736	56	972	39	1248	63
200	31	550	62	768	42	1026	66	1275	93
225	31	576	28	780	60	1040	73	1280	73
234	32	588	57	784	57	1064	49	1296	91

Are there infinitely many census-taker numbers? In particular, there would be if we knew that there are infinitely many primes p such that $2p-1$ is also prime. This seems likely, but a proof (or disproof) seems to be beyond present-day techniques.

A problem somewhat dual to the census-taker problem was investigated by Kelly [8], who showed that for every integer M exceeding 18 there are distinct triples with the sum M and equal (unspecified) products. The parallel is not exact, however, since in the census-taker problem it is not enough to find equal sums for a given product. We must find triples with *exactly one pair* of equal sums, and this added feature is what makes the problem both tricky and interesting.

We thank a referee for supplying several references.

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How Often Should You Beat Your Kids?

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A result is proved which shows, roughly speaking, that one should beat one's kids every day except Sunday.

This note is a follow-up to the note "How to Beat Your Kids at Their Own Game," by K. Levasseur [1], in which the author proposes the following game to be played against one's two-year-old children: Starting with a deck consisting of n red cards and n black cards (in typical applications, $n = 26$), the cards are turned up one at a time, each player at each stage predicting the color of the card which is about to appear. The kid is supposed to guess "Red" or "Black" randomly with equal probability (this solves the problem of constructing a perfect random number generator), while you play what is obviously the optimal strategy—guessing randomly (or, if you prefer, always saying "Black") whenever equal numbers of cards of both colors remain in the deck and otherwise predicting the color which is currently in the majority. Levasseur analyzes the game and shows that on the average you will have a score of $n + (\sqrt{\pi n} - 1)/2 + O(n^{-1/2})$, while the kid, of course, will have an average score of exactly n.

We, however, maintain that only the most degenerate parent would play against a two-year-old for money, and that our concern must therefore be, not *by how much* you can expect to win, but with what probability you will win *at all*. Our principal result is that this probability tends asymptotically to 85.4% (more precisely: to $1/2 + 1/\sqrt{8}$) as n tends to infinity. This shows with what unerring instinct Levasseur's mother selected the game—the high 85% loss rate will instill in the young progeny a due respect for the immense superiority of their parents, while the 15% win rate will maintain their interest and prevent them from succumbing to feelings of hopelessness and frustration.

The analysis begins as in Levasseur's article: each of the $\binom{2n}{n}$ possible orderings of the cards into red and black elements corresponds to a path p moving downwards and leftwards from an initial value $(R, B) = (n, n)$ to a final value $(R, B) = (0, 0)$ of the pair (R, B), where R and B denote the numbers of red and black cards remaining, respectively. If this path meets the diagonal $R = B$ a total of $m(p)$ times, where the initial point at (n, n) is counted but the final point at $(0, 0)$ is not, then the expected win of the parent is $m(p)/2$. Indeed, at each meeting point the parent guesses randomly, with an expected score of $1/2$ and hence an expected win over his child of 0; between each pair of meeting points, the parent will consistently guess "Red" or consistently "Black," depending on whether p is now below or above the diagonal, and will be right exactly one more time than he is wrong, gaining exactly half a point over his randomly guessing child. Levasseur shows that the average value of $m(p)$, as p ranges over the set $\mathcal{P}_n$ of paths as described above, is exactly $4^n/\binom{2n}{n} - 1$, leading to the result on the expected win stated above. To solve the problem we have set ourselves, we must answer two questions:

—(i) for a given value of $m(p)$, what is the probability of winning? and

—(ii) with what probability will $m(p)$ take on a given value m, $1 \leqslant m \leqslant n$?

We answer the second question first.

Let $N_m(n)$ denote the number of paths $p \in \mathscr{P}_n$ with $m(p) = m$. For $n = 0$ this equals 1 if $m = 0$ and 0 otherwise, but for positive n we must have $m \geqslant 1$ since the initial point of the path is counted as a meeting with the diagonal. If a path $p \in \mathscr{P}_n$ meets the diagonal more than once, i.e., if $m(p) > 1$, then the first meeting point will be at some value (k, k) of (R, B) with $1 \leqslant k \leqslant n-1$. Conversely, if we pick such a k, then the number of paths $p \in \mathscr{P}_n$ with $m(p) = m$ and having (k, k) as their first meeting point will be equal to the product of $N_1(n-k)$ (the number of ways of descending from (n, n) to (k, k) without meeting the diagonal on the way) and $N_{m-1}(k)$ (the number of ways of descending from (k, k) to $(0,0)$ with exactly $m-1$ further meetings). Hence

$$N_m(n) = \sum_{k=1}^{n-1} N_1(n-k) N_{m-1}(k) \qquad (m > 1).$$

It follows that the generating function $\mathscr{N}_m(x) = \sum_{n=m}^{\infty} N_m(n) x^n$ is the product of $\mathscr{N}_1(x)$ and $\mathscr{N}_{m-1}(x)$, and hence that $\mathscr{N}_m(x) = \mathscr{N}_1(x)^m$. This formula holds also for $m = 0$ since $N_0(n) = 0$ for all positive n. On the other hand, the sum of all the functions $\mathscr{N}_m(x)$ is the generating function whose nth coefficient is the total number of paths in $\mathscr{P}_n$, i.e., $\binom{2n}{n}$. Hence

$$\frac{1}{1-\mathscr{N}_1(x)} = \sum_{m=0}^{\infty} \mathscr{N}_1(x)^m = \sum_{m=0}^{\infty} \mathscr{N}_m(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

so

$$\mathscr{N}_1(x) = 1 - \sqrt{1-4x}, \qquad \mathscr{N}_m(x) = \left(1-\sqrt{1-4x}\right)^m.$$

Using the well-known Taylor expansion of this function, we find:

$$N_m(n) = 2^m \cdot \frac{m}{n} \cdot \binom{2n-m-1}{n-1} \qquad (1 \leqslant m \leqslant n).$$

Therefore, the probability for a random path $p \in \mathscr{P}_n$ to have $m(p) = m$ is given by

$$\operatorname{prob}\{\, m(p) = m \,\} = \frac{N_m(n)}{\binom{2n}{n}} = \frac{m}{2n} \cdot \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)}{\left(1-\frac{1}{2n}\right)\left(1-\frac{2}{2n}\right)\cdots\left(1-\frac{m}{2n}\right)}.$$

For m of the order of $\sqrt{n}$ (the right order according to Levasseur's analysis), this will

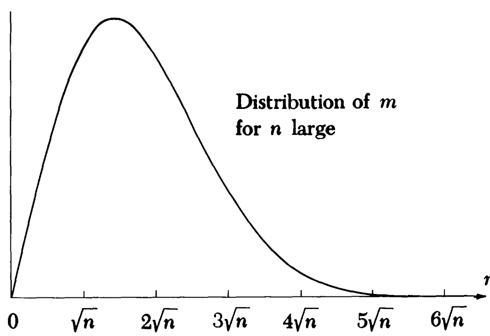

FIGURE 1

be asymptotically equal to $(m/2n)e^{-m^2/4n}$ (cf. FIGURE 1). As a test, when n is large we have for the total probability

$$\sum_{m=0}^{\infty} \text{prob}\{m(p)=m\} \approx \sum_{m=0}^{\infty} \frac{m}{2n} e^{-m^2/4n}$$

$$\approx \int_0^{\infty} \frac{x}{2n} e^{-x^2/4n}\,dx = -e^{-x^2/4n}\Big|_0^{\infty} = 1$$

and for the expected value of m the value

$$\sum_{m=0}^{\infty} \text{prob}\{m(p)=m\}\cdot m \approx \sum_{m=0}^{\infty} m\frac{m}{2n} e^{-m^2/4n}$$

$$\approx \int_0^{\infty} x\frac{x}{2n} e^{-x^2/4n}\,dx = 4\sqrt{n}\int_0^{\infty} t^2 e^{-t^2}\,dt = \sqrt{\pi n},$$

in accordance with Levasseur's result.

We now turn to the first of the two questions above. For the reasons already explained, for an ordering of cards given by a path $p \in \mathscr{P}_n$ with $m(p)=m$, of the $2n-m$ turns corresponding to points on p not on the diagonal one will guess correctly exactly n times and incorrectly exactly $n-m$ times, while the probability of guessing correctly at one of the m turns corresponding to points on the diagonal is 50% each time. Hence one's total number of correct guesses will be described by a bell-shaped curve centered around the expected value $n+\frac{1}{2}m$ and with a width of the order of $\sqrt{m}$, or (for almost all paths p) $\sqrt[4]{n}$ (cf. FIGURE 2). On the other hand, if one guesses correctly $n+k$ times, then one's chance of beating the randomly playing kid is

$$\frac{1}{2^{2n}}\sum_{r=0}^{n+k-1}\binom{2n}{r} \approx \frac{1}{2} + 2^{-2n}\sum_{r=0}^{k}\binom{2n}{n+r},$$

and since $2^{-2n}\binom{2n}{n+r} \approx \sqrt{1/\pi n}\, e^{-r^2/n}$ by Stirling's formula, this is approximately equal to

$$\frac{1}{2} + \frac{1}{\sqrt{\pi n}}\sum_{r=0}^{k} e^{-r^2/n} \approx \frac{1}{2} + \frac{1}{\sqrt{\pi n}}\int_0^k e^{-u^2/n}\,du = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{k/\sqrt{n}} e^{-u^2}\,du.$$

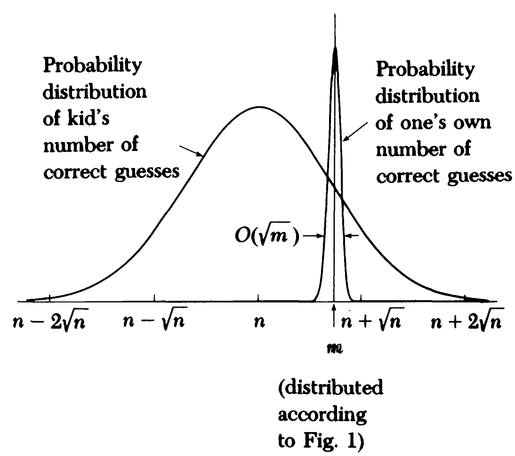

FIGURE 2

Since $k/\sqrt{n}$ is almost always very near to $\frac{1}{2}m/\sqrt{n}$, the probability of winning when $m(p)=m$ is very nearly equal to

$$\frac{1}{2}+\frac{1}{\sqrt{\pi}}\int_0^{m/2\sqrt{n}} e^{-u^2}\,du.$$

Multiplying this by the probability that $m(p)=m$ as computed above, we find finally

$$\begin{aligned}
\text{probability of winning} &\approx \frac{1}{2}+\sum_{m=0}^{\infty}\frac{m}{2n}e^{-m^2/4n}\left(\frac{1}{\sqrt{\pi}}\int_0^{m/2\sqrt{n}} e^{-u^2}\,du\right)\\
&\approx \frac{1}{2}+\int_0^{\infty}\frac{x}{2n}e^{-x^2/4n}\left(\frac{1}{\sqrt{\pi}}\int_0^{x/2\sqrt{n}} e^{-u^2}\,du\right)dx\\
&= \frac{1}{2}+\frac{1}{2\sqrt{\pi}}\int_0^{\infty} xe^{-x^2/4}\left(\int_0^{x/2} e^{-u^2}\,du\right)dx\\
&= \frac{1}{2}+\frac{1}{2\sqrt{\pi}}\int_0^{\infty} e^{-u^2}\left(\int_{2u}^{\infty} xe^{-x^2/4}\,dx\right)du\\
&= \frac{1}{2}+\frac{1}{2\sqrt{\pi}}\int_0^{\infty} e^{-u^2}\left(2e^{-u^2}\right)du\\
&= \frac{1}{2}+\frac{1}{2\sqrt{2}},
\end{aligned}$$

as claimed. This is very nearly 6/7, so the result of our paper can be conveniently implemented by beating one's kids on weekdays and Saturdays, but never on Sunday.

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A Note on the Five-Circle Theorem

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In his paper [1] H. Demir stated and proved

THE FIVE-CIRCLE THEOREM. *Let P and Q be two points on the side BC of a triangle ABC in the order B, P, Q, C. If the triangles ABP, APQ, AQC have congruent incircles, then the triangles ABQ, APC have congruent incircles.*

He also asked for a geometric proof of this theorem.

Here we give such a proof for the following more general

FOUR-CIRCLE THEOREM. *Let P and Q be two points on the side BC of a triangle ABC. Then the triangles ABP and AQC have congruent incircles if and only if the triangles ABQ and APC have congruent incircles.*

Proof. We omit the trivial case when P coincides with Q. Without loss of generality we may assume that P lies between B and Q (see FIGURE 1). Denote by r, r_1, r_2, ρ_1 and ρ_2 the radii of the incircles k, k_1, k_2, k' and k'', respectively, of the triangles APQ, ABP, AQC, ABQ and APC. Let T, T_1 and T_q be the tangency points of the line BC, respectively, with k, k_1 and the excircle k_q of the triangle APQ (see FIGURE 2).

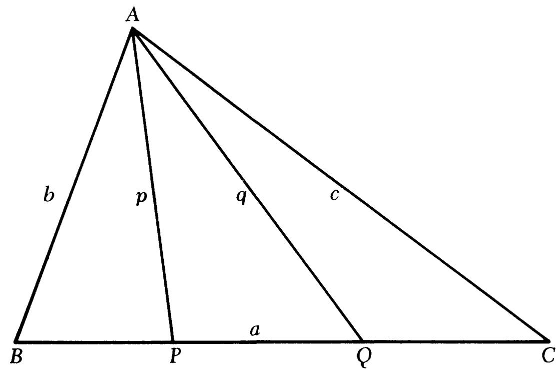

FIGURE 1

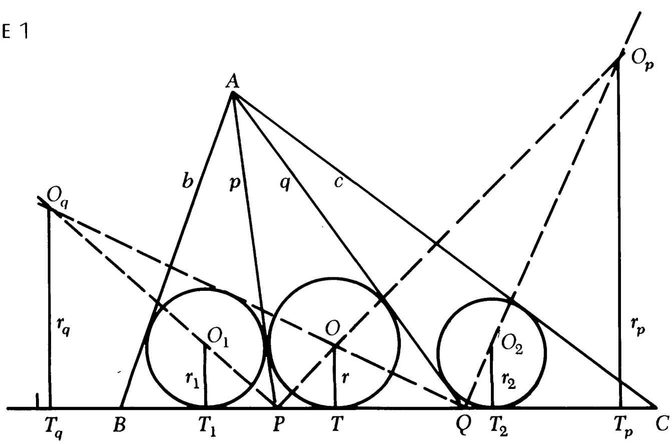

FIGURE 2

From similar triangles, we obtain

$$\frac{r}{r_q} = \frac{QT}{QT_q} \quad \text{and} \quad \frac{r_1}{r_q} = \frac{PT_1}{PT_q},$$

where r_q is the radius of k_q. Therefore,

$$\frac{r}{r_1} = \frac{QT}{QT_q} \cdot \frac{PT_q}{PT_1}. \tag{1}$$

For convenience denote (see FIGURE 1) AB, AP, AQ, AC and PQ by b, p, q, c and a, and the semiperimeters of triangles ABP, APQ and AQC, respectively, by s_1, s and s_2. Then (see [2, p. 87]) $QT = s - p$, $QT_q = s$, $PT_q = QT_q - PQ = s - a$ and $PT_1 = s_1 - b$, and (1) becomes

$$\frac{r}{r_1} = \frac{s-p}{s} \cdot \frac{s-a}{s_1-b}.$$

Similarly, we obtain

$$\frac{r}{r_2} = \frac{s-q}{s} \cdot \frac{s-a}{s_2-c}.$$

Hence r_1 and r_2 are equal if and only if

$$(s-p)(s_2-c) = (s-q)(s_1-b). \tag{2}$$

In a similar way, let T' and T'' denote the tangency points of BC with k' and k'' (see FIGURE 3). From similar triangles, we obtain

$$\frac{\rho_1}{r} = \frac{QT'}{QT} \quad \text{and} \quad \frac{\rho_2}{r} = \frac{PT''}{PT};$$

therefore, $\rho_1 = \rho_2$ if and only if $PT \cdot QT' = QT \cdot PT''$, that is, if and only if

$$(s-q)(\sigma_1-b) = (s-p)(\sigma_2-c), \tag{3}$$

where σ_1 and σ_2 are the semiperimeters of triangles ABQ and APC, respectively.

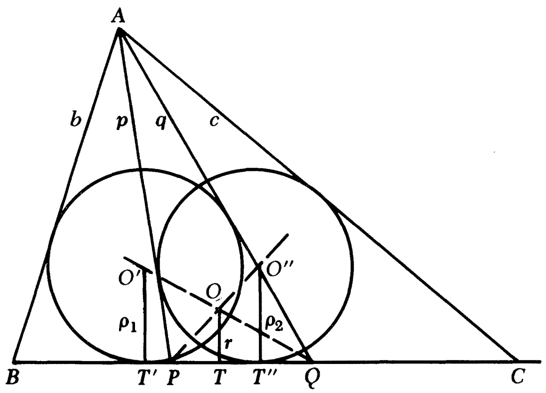

FIGURE 3

But, clearly, $\sigma_1 = s + s_1 - p$ and $\sigma_2 = s + s_2 - q$; hence

$$\begin{aligned}(s-q)&(\sigma_1-b) - (s-p)(\sigma_2-c)\\ &= (s-q)(s+s_1-p-b) - (s-p)(s+s_2-q-c)\\ &= (s-q)(s-p) + (s-q)(s_1-b) - (s-p)(s-q) - (s-p)(s_2-c)\\ &= (s-q)(s_1-b) - (s-p)(s_2-c).\end{aligned}$$

Consequently (2) and (3) are equivalent, which proves the theorem.

The Four-Circle Theorem can be easily proved also from Demir's equation (4) in the article [1], where this equation is obtained by means of trigonometry.

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Elementary Proof That Some Angles Cannot Be Trisected by Ruler and Compass

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Prefatory note, May 1987

This purely expository paper dates from April 1946. Robert Lee Durham, president emeritus of Southern Seminary and Junior College, had sent me a hundred dollars and asked me to make it clear to him why an angle cannot in general be trisected by ruler and compass. He had himself presented a way of almost trisecting any angle by ruler and compass, to an accuracy for acute angles of 1/720 of a degree; see his "Simple construction for the approximate trisection of an angle," *Amer. Math. Monthly* 51 (1944), 217–218.

I welcomed the money and the occasion to familiarize myself with the famous proof. I was guided in large part by L. E. Dickson, "Why it is impossible to trisect an angle or construct a regular polygon of 7 or 9 sides by ruler and compass," *Mathematics Teacher* 14 (1921), 217–223.

Mr. Durham expressed satisfaction with my report and proposed paying for publishing it as a pamphlet. With his approval I submitted it instead to a mathematics journal. After waiting nineteen months for a decision from the journal, I recalled the paper and dropped the matter.

Michael Stueben learned of this from my autobiography (*The Time of My Life*, M.I.T., 1985) and expressed interest. I sent him a copy of the paper. He suggested I ask Professor Underwood Dudley whether there was any point in publishing it. I had no preconception in the matter. Professor Dudley put Professor Alexanderson onto it, and I am pleased and flattered by Professor Alexanderson's proposal to publish it.

The proof that some angles cannot be trisected by ruler and compass is presented here in a way which departs from previous formulations in two main respects: (1) no mathematical principles beyond the scope of first courses in algebra and plane geometry are appealed to, unless proved in the course of the paper; (2) especial attention is accorded to rigor and detail in proving the lemma which relates ruler-and-compass constructions to the arithmetical operations of addition, subtraction, multiplication, division, and square root.

I. In constructions by ruler and compass, points are determined by intersections of lines with lines, lines with circles, or circles with circles. But the construction of a line or a circle rests in turn upon previously constructed or initially assumed points; for a line is drawn through two given points, and a circle is described about a given point using as radius the interval between two given points. Hence we may without loss of generality speak of the construction of points as construction from previously given *points*. The construction of a point from given points may proceed in any of three ways:

(i) by drawing a line through two of the given points, and another line through two others, and taking their intersection;

(ii) by drawing a line through two of the given points, and a circle around some one of the given points with radius equal to the interval between two of the given points, and taking one of the intersections of the line with the circle;

(iii) by drawing a circle as in (ii), and another circle similarly, and taking one of the intersections of the two circles.

These are the three possible forms of what may be called the *immediate* construction of a point from given points. Any other construction of a point from given points (by ruler and compass) will consist of a succession of such immediate constructions, using, at each stage, any of the points which have meanwhile been constructed in the course of the succession of immediate constructions.

Certain constructions, from two given points A and B, give points whose distances from the line AB can be simply expressed in terms of the length AB itself. E.g., let Q be constructed from A and B as an intersection of circles described around A and around B with radius equal to the interval AB. Since the triangle ABQ is equilateral, a perpendicular QR dropped from Q to AB divides AB into equal parts AR and RB each of which is half as long as BQ. By the Pythagorean theorem, the square of the length BQ is the sum of the squares of the lengths RB and RQ. From these data it is readily calculated that the perpendicular distance (RQ) of Q from AB is $\sqrt{3}/2$ times the length AB. Also the perpendicular distance of Q from a line AY perpendicular to AB at A is easily expressible in terms of the length AB; namely, as half of it.

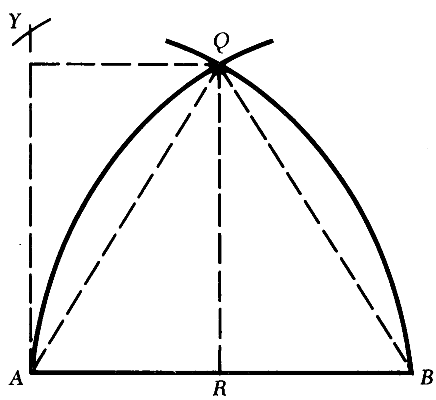

FIGURE 1

No units of length having been chosen, we are free to fix it as we wish. Let us take our unit as half the length AB. Then the distance of Q from AB becomes $\sqrt{3}$, and the distance of Q from AY becomes 1.

Let us call the distance of any point from the line AB the *ordinate* of the point (measured in terms of the unit above specified), and let us call the distance from the line AY the ***abscissa***. Thus the abscissa of Q is 1; the ordinate of Q is $\sqrt{3}$; the abscissa of B is 2; and the ordinate of B and the abscissa and ordinate of A are all 0. If a point is below AB its ordinate will be given a minus sign, and if it is to the left of AY its abscissa will be given a minus sign.

Now let us define a certain class α of points as follows. A point belongs to α just in case both its abscissa and its ordinate are expressible in terms of integers and the operations of addition, subtraction, multiplication, division, and square root. Thus Q, for example, belongs to α; and so do A and B.

It will now be proved that any point P which is constructed by ruler and compass *immediately* from points belonging to α must itself belong to α. There are three cases to consider, corresponding to (i)–(iii) above.

Case 1. P is the intersection of two lines, drawn respectively through points S and T belonging to α and points U and V belonging to α. Let p, s, t, u, and v be the respective abscissas of P, S, T, U, and V, and let p', s', t', u', and v' be the respective ordinates. Then, where SX_1 and SX_2 are drawn parallel to AB, and PX_1 and TX_2 are drawn parallel to AY, the lengths SX_1, SX_2, X_1P, and X_2T will be equal to the respective differences $p-s$, $t-s$, $p'-s'$, and $t'-s'$; so, by similarity of the triangles SPX_1 and STX_2, we get the proportion

$$\frac{p-s}{t-s}=\frac{p'-s'}{t'-s'}$$

which, solved for p, gives

$$p=\frac{st'-s't-p'(s-t)}{t'-s'}. \tag{1}$$

Similar reasoning, in terms of U and V instead of S and T, gives

$$p=\frac{uv'-u'v-p'(u-v)}{v'-u'}.$$

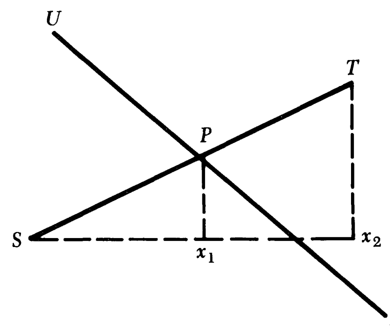

FIGURE 2

Combining this result with (1), we get

$$\frac{st'-s't-p'(s-t)}{t'-s'}=\frac{uv'-u'v-p'(u-v)}{v'-u'}.$$

Solving this for p', we obtain

$$p'=\frac{(st'-s't)(v'-u')-(uv'-u'v)(t'-s')}{(s-t)(v'-u')-(t'-s')(u-v)}. \tag{2}$$

But, since S, T, U, and V belong to α, we can express s, s', t, t', u, u', v, and v' in terms of integers and addition, subtraction, multiplication, division, and square root. The same is true of p', since, by (2), we can express p' in terms of s, s', t, t', u, u', v, and v' and subtraction, multiplication, and division. Hence the same is true in turn of p, in view of (1). Therefore, P belongs to α.

Case 2. P is an intersection of a line, drawn through two points S and T belonging to α, with a circle described about some point U belonging to α; and the radius of the circle is equal to the distance between two points V and W belonging to α. By the Pythagorean theorem, the square of the distance PU can be expressed in terms of the horizontal distance $p-u$ and the vertical distance $p'-u'$ as $(p-u)^2+(p'-u')^2$ (where the small letters represent abscissas and the accented small letters represent ordinates as in Case 1). Likewise the square of the distance VW is $(w-v)^2+(w'-v')^2$. Then, since the distances PU and VW are the same,

$$(p-u)^2+(p'-u')^2=(w-v)^2+(w'-v')^2. \tag{3}$$

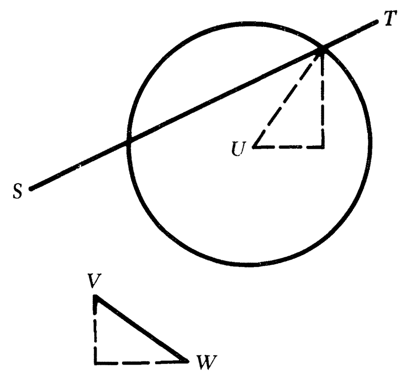

FIGURE 3

Also (1) still holds as in Case 1; so, substituting in (3) according to (1), we get

$$\left(\frac{st'-s't-p'(s-t)}{t'-s'}-u\right)^2+(p'-u')^2=(w-v)^2+(w'-v')^2.$$

We can expand this and then solve it for p' by the familiar formula for solving quadratic equations. We thus get an expression for p' built up of s, s', t, t', u, etc. by means of addition, subtraction, multiplication, division, and square root.* But s, s', t, etc. are themselves expressible by means of those same operations on the basis of integers; so the same is true of p'. Hence, by (1), the same is true also of p. So P belongs to α.

Case 3. P is an intersection of two circles; one of the circles is described about some point Z, belonging to α, with radius equal to the distance between two points S and T belonging to α; and the other circle is as in Case 2. So now we have not only (3) again, as in Case 2, but also, by the same reasoning,

$$(p-z)^2+(p'-z')^2=(t-s)^2+(t'-s')^2. \tag{4}$$

Expanding the left sides of (3) and (4), and then subtracting the one equation from the other, we get:

$$\begin{aligned}-2pz+2pu+z^2-u^2-2p'z'+2p'u'+z'^2-u'^2\\ =(t-s)^2-(w-v)^2+(t'-s')^2-(w'-v')^2.\end{aligned}$$

Therefore,

*Integral coefficients and exponents would also turn up, but they are inessential; for example, $4x$ is $x+x+x+x$, and x^2 is xx.

$$p=\frac{(t-s)^2-(w-v)^2+(t'-s')^2-(w'-v')^2-z^2+u^2+2p'z'-2p'u'-z'^2+u'^2}{2u-2z}. \tag{5}$$

The rest of the argument is like the argument of Case 2 from (3) onward, except that we use (5) now instead of (1).

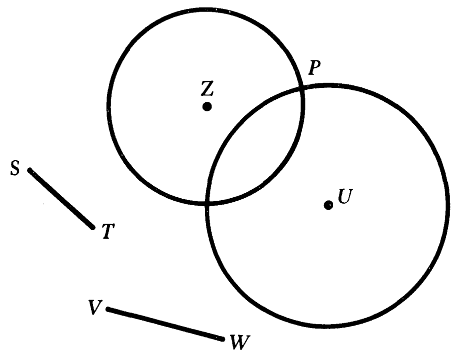

FIGURE 4

In each of the three cases, therefore, P must belong to α.

Now, starting with the same old points A and B, let us make the following constructions with ruler and compass. Describe a circle around A with radius AB, obtaining C as the other intersection of the circle with the line BA produced. Describe a circle of same radius around C, obtaining D as an intersection of the two circles.

Next let us *assume that the angle DAB can be trisected by ruler and compass*. This means we can obtain, by ruler and compass, points E and F such that the angles BAF, FAE, and EAD are equal. Finally drop a perpendicular from B upon AF (which can be done by ruler and compass in several steps in familiar fashion), meeting AF at G.

Now observe that C belongs to α; for, A and B belong to α, and we have seen that any point P constructed by ruler and compass immediately from points belonging to α must also belong to α. It follows then also that D belongs to α, since D was constructed immediately from the members C and A of α; correspondingly for each of the succeeding points of our supposed construction, including finally G. Thus, *if*

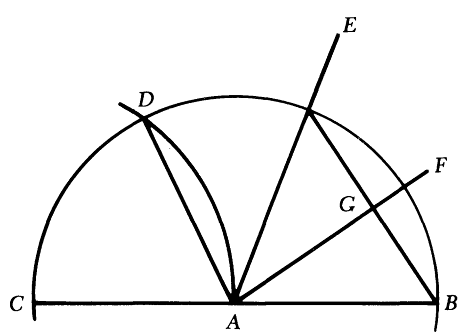

FIGURE 5

the supposed construction was possible, G belongs to α. This means that the abscissa g and ordinate g' of G (distance to the right of A and distance above A) are expressible in terms of integers by addition, subtraction, multiplication, division, and square root. Then *the distance from A to G (let us call it m) must be expressible in terms of integers by addition, subtraction, multiplication, division, and square root*, since it is equal to $\sqrt{g^2+g'^2}$ by the Pythagorean theorem.

II. In this section a further fact will be established concerning the quantity m (distance from A to G in FIGURE 5), namely, that $m^3=3m-1$.

Let us copy FIGURE 5 (leaving out the circles, to reduce cluttering) and introduce further constructions as follows (FIGURE 6): erect a perpendicular to AB at B, meeting AF at H; erect a perpendicular to AF at H, meeting AE at I; erect a perpendicular to AE at I, meeting AD (produced) at J; and drop a perpendicular from J to AC (produced), meeting it at K. Finally drop perpendiculars from I to JK and AB, meeting them respectively in L and M; and drop a perpendicular from H to IM, meeting it in N.

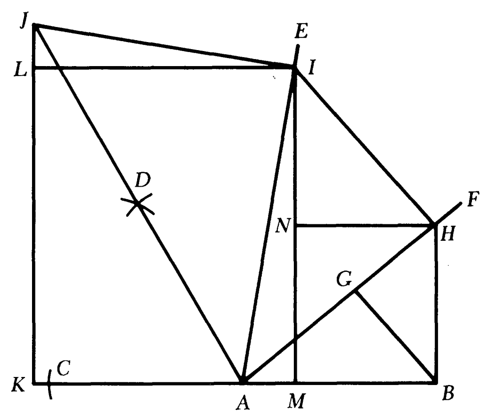

FIGURE 6

Let us represent distances between points by pairing the letters representing the points and applying square brackets; thus

$$[AB]=2, \qquad [AG]=m \tag{6}$$

(cf. Sec I). By (6) and the Pythagorean theorem, further,

$$[BG]=\sqrt{4-m^2}. \tag{7}$$

Distances expressed by square brackets are to be understood always as the appropriate positive quantities, ignoring direction; the distinction between positive and negative abscissas and ordinates made in Section I will not be needed again, nor will there be any further reference to abscissas and ordinates at all. Note in particular that the letter m, arbitrarily chosen for $[AG]$, has nothing to do with the point M.

Since the angles JAI, IAH and HAB in FIGURE 6 are all equal (by the assumed construction of FIGURE 5), the four right triangles JAI, IAH, HAB, and BAG all have an acute angle equal and are therefore similar. The sides of the triangles JAI, IAH, and HAB are therefore proportional to those of BAG, so that from (6)–(7) we can infer

$$\frac{[AI]}{[AJ]} = \frac{[AH]}{[AI]} = \frac{[AB]}{[AH]} = \frac{m}{2}, \tag{8}$$

$$\frac{[IJ]}{[AJ]} = \frac{[HI]}{[AI]} = \frac{[BH]}{[AH]} = \frac{\sqrt{4-m^2}}{2}. \tag{9}$$

By (6) we can put 2 for $[AB]$ in (8), getting

$$[AH] = \frac{4}{m}. \tag{10}$$

Accordingly, putting $4/m$ for $[AH]$ in (8), we get

$$[AI] = \frac{8}{m^2}. \tag{11}$$

Accordingly, putting $8/m^2$ for $[AI]$ in (8), we get

$$[AJ] = \frac{16}{m^3}. \tag{12}$$

Substituting these recovered values for $[AH]$, $[AI]$, and $[AJ]$ in (9), we get

$$[IJ] = \frac{8\sqrt{4-m^2}}{m^3}, \tag{13}$$

$$[HI] = \frac{4\sqrt{4-m^2}}{m^2}, \tag{14}$$

$$[BH] = \frac{2\sqrt{4-m^2}}{m}. \tag{15}$$

The triangles HIN and HAB are similar, having homologous sides perpendicular each to each; therefore, their sides are proportional:

$$\frac{[IN]}{[HI]} = \frac{[AB]}{[AH]} = \frac{m}{2}. \quad \text{(cf. (8))}$$

Hence, by (14),

$$[IN] = \frac{2\sqrt{4-m^2}}{m}.$$

By (15), then, since $[IM] = [BH] + [IN]$ (see FIGURE 6),

$$[IM] = \frac{4\sqrt{4-m^2}}{m}. \tag{16}$$

By the Pythagorean theorem, $[AI]^2 = [AM]^2 + [IM]^2$; hence, by (11) and (16),

$$\frac{64}{m^4} = [AM]^2 + \frac{16(4-m^2)}{m^2},$$

whose solutions are

$$[AM] = \pm\left(4 - \frac{8}{m^2}\right). \tag{17}$$

Since $[AM]$ must not be negative, the sign "$\pm$" in (17) should be resolved as plus or as minus according as the quantity in parentheses is itself positive or negative; and we shall soon see which it is.

The angle KAJ of FIGURE 6 is the angle CAD of FIGURE 5; and this was constructed like the angle ABQ of FIGURE 1, and is hence equal to it. Therefore, just as $[RB]$ in FIGURE 1 was seen (Section I) to be half of $[BQ]$, so $[AK]$ in FIGURE 6 is half of $[AJ]$. Hence, by (12),

$$[AK] = \frac{8}{m^3}.$$

By this and (17), then, since $[IL] = [AK] + [AM]$ (see FIGURE 6),

$$[IL] = \frac{8}{m^3} \pm \left(4 - \frac{8}{m^2}\right). \tag{18}$$

The triangles ILJ and IMA are similar, having homologous sides perpendicular each to each; therefore, their sides are proportional:

$$\frac{[IL]}{[IJ]} = \frac{[IM]}{[AI]}.$$

Substituting in this from (18), (13), (16), and (11) and reducing, we get

$$2 \pm (m^3 - 2m) = 4m - m^3, \tag{19}$$

where "$\pm$" is plus or minus according to whether it is plus or minus in (17). Now if the correct reading were minus, (19) would reduce to "$m = 1$" and hence (17) would become "$[AM] = 4$" and (11) would become "$[AI] = 8$"; accordingly $[AM]/[AI]$ would equal $[AG]/[AB]$, and this would imply identity of the angles MAI and GAB, which is impossible, since those angles were originally assumed to be constructed so as to be respectively two thirds and one third of the angle DAB of FIGURE 5. Therefore, the correct reading of "$\pm$" in (19) is plus, so (19) reduces to

$$m^3 = 3m - 1 \tag{20}$$

q.e.d.

III. We saw in Section I (on the assumption that the angle DAB could be trisected by ruler and compass) that m must be expressible in terms of integers by addition, subtraction, multiplication, division, and square root. It will now be proved that the use of square root is necessary, i.e., that m cannot be expressed in terms of integers by mere addition, subtraction, multiplication, and division. Suppose the contrary: that m could be so expressed. Then we could carry out all the operations and reduce to lowest terms, getting a simple fraction

$$m = \frac{n}{d},$$

where n and d are positive or negative integers without common factor above 1. So then, by (20), we should have

$$\frac{n^3}{d} = 3nd - d^2, \tag{21}$$

showing that n^3/d is whole and hence that d is a factor of n^3. But, since d and n

have no common factor above 1, there can be no factor above 1 common to d and n^3; so, in order to be a factor of n^3, d must itself be 1 or -1. By (21), then,

$$n^3 = 3n - 1 \text{ or } -n^3 = -3n - 1,$$

i.e.,

$$n(n^2 - 3) = \pm 1.$$

Then, since n and $n^2 - 3$ are integers, n must be ± 1 and $n^2 - 3$ must be ± 1. But *this is impossible, since, where* $n = \pm 1$, $n^2 - 3 = -2$. *So the assumption that* m *can* be expressed without use of square root is disproved. We have found that the expression of m must use radicals.

If a radical contains a radical within it, e.g. "$\sqrt{5+\sqrt{3}}$", it will be called of *order* 2; if it contains radicals three deep, e.g. "$\sqrt{5+\sqrt{\sqrt{5}+5}}$", it will be called of order 3; and so on. If it contains no radical within it, e.g., "$\sqrt{5}$", it will be called of order 1. Now let us suppose the expression of m reduced as far as possible in the following way: if some radical (call it ρ) occurring in the expression of m is expressible in another way, employing no radicals of the order of ρ except perhaps some which already occur elsewhere in the given expression of m, then let ρ be supplanted by that equivalent expression. Carrying this type of reduction as far as possible, we end up with an expression ζ for m which has this property: *none of the radicals in it is expressible in terms of integers, addition, subtraction, multiplication, division, lower-order radicals, and equal-order radicals recurring elsewhere in* ζ.

Now let $\sqrt{\vartheta}$ be any one of the highest-order radicals appearing in ζ, and let us examine two possible cases, according as $\sqrt{\vartheta}$ does not or does put in an appearance under a fraction-line in ζ.

Case 1: it does not. Then ζ consists of an algebraic sum of terms some of which exhibit $\sqrt{\vartheta}$ as an explicit factor once each (not more, because $\sqrt{\vartheta}\cdot\sqrt{\vartheta}$ becomes ϑ) and the rest of which does not involve $\sqrt{\vartheta}$ at all. Collecting the latter terms as σ (if there is none, $\sigma = 0$) and collecting the former ones (with $\sqrt{\vartheta}$ deleted) as τ, we can express ζ as $\sigma + \tau\sqrt{\vartheta}$.

Case 2: it does. Then reduce all of ζ to a common denominator. The numerator and denominator are each similar to Case 1, so that ζ can be expressed in the form

$$\zeta = \frac{\sigma' + \tau'\sqrt{\vartheta}}{\sigma'' + \tau''\sqrt{\vartheta}}$$

(where, however, τ' as well as σ' and σ'' may be 0). Multiplying numerator and denominator by $\sigma'' - \tau''\sqrt{\vartheta}$, we get

$$\zeta = \frac{\sigma'\sigma'' - \tau'\tau''\vartheta + (\sigma''\tau' - \sigma'\tau'')\sqrt{\vartheta}}{\sigma''^2 - \tau''^2\vartheta}.$$

Taking σ and τ then, respectively, as

$$\frac{\sigma'\sigma'' - \tau'\tau''\vartheta}{\sigma''^2 - \tau''^2\vartheta} \quad\text{and}\quad \frac{\sigma''\tau' - \sigma'\tau''}{\sigma''^2 - \tau''^2\vartheta},$$

we again get an expression of the form $\sigma + \tau\sqrt{\vartheta}$ for ζ.

We see, therefore, that ζ is expressible, in either of the two possible cases, in the form

$$\zeta = \sigma + \tau\sqrt{\vartheta}\,. \tag{22}$$

Then, by (20),

$$(\sigma + \tau\sqrt{\vartheta})^3 = 3(\sigma + \tau\sqrt{\vartheta}) - 1. \tag{23}$$

Expanding and rearranging this, we get

$$\sigma^3 + 3\sigma\tau^2\vartheta - 3\sigma + 1 + (3\sigma^2\tau + \tau^3\vartheta - 3\tau)\sqrt{\vartheta} = 0. \tag{24}$$

Now the coefficient of $\sqrt{\vartheta}$ here, viz. $3\sigma^2\tau + \tau^3\vartheta - 3\tau$, must be equal to 0; for, if it were not, we could divide equation (24) by it, getting

$$\sqrt{\vartheta} = -\frac{\sigma^3 + 3\sigma\tau^2\vartheta - 3\sigma + 1}{3\sigma^2\tau + \tau^3\vartheta - 3\tau},$$

and this would mean, contrary to the earlier italicized description of ζ, that $\sqrt{\vartheta}$ was expressible using no radicals except the other radicals already occurring in ζ. So

$$3\sigma^2\tau + \tau^3\vartheta - 3\tau = 0. \tag{25}$$

Accordingly this portion vanishes from (24), leaving

$$\sigma^3 + 3\sigma\tau^2\vartheta - 3\sigma + 1 = 0,$$

which is algebraically transformable into

$$\sigma(3\sigma^2 + \tau^2\vartheta - 3) + 2\sigma\tau^2\vartheta - 2\sigma^3 + 1 = 0. \tag{26}$$

We know that $\tau \neq 0$, since otherwise, in view of (22), the radical $\sqrt{\vartheta}$ would have been superfluous. So we can divide (25) by τ, getting

$$3\sigma^2 + \tau^2\vartheta - 3 = 0. \tag{27}$$

Hence (26) becomes

$$2\sigma\tau^2\vartheta - 2\sigma^3 + 1 = 0,$$

i.e.,

$$2\sigma^3 - 2\sigma\tau^2\vartheta = 1. \tag{28}$$

By (27),

$$3\sigma^2 + \tau^2\vartheta = 3. \tag{29}$$

Now for *any* x, as can be seen by multiplying out,

$$x^3 - (3\sigma^2 + \tau^2\vartheta)x + 2\sigma^3 - 2\sigma\tau^2\vartheta = (x - \sigma - \tau\sqrt{\vartheta})(x - \sigma + \tau\sqrt{\vartheta})(x + 2\sigma).$$

Substituting in this from (28) and (29), we have

$$x^3 - 3x + 1 = (x - \sigma - \tau\sqrt{\vartheta})(x - \sigma + \tau\sqrt{\vartheta})(x + 2\sigma). \tag{30}$$

Taking x then in particular as -2σ, we get

$$(-2\sigma)^2 - 3(-2\sigma) + 1 = (-3\sigma - \tau\sqrt{\vartheta})(-3\sigma + \tau\sqrt{\vartheta}) \cdot 0 = 0,$$

i.e.,

$$(-2\sigma)^3 = 3(-2\sigma) - 1. \tag{31}$$

At the beginning of this section it was proved, from (20), that m needs radicals in its expression. By repeating that argument now we can conclude from the present result (31) (which is similar in form to (20)) that -2σ must contain radicals.

Let $\sqrt{\eta}$, then, be any one of the radicals of highest order in -2σ. Just as we put ζ into the form $\sigma + \tau\sqrt{\vartheta}$, so we can put -2σ into a corresponding form $\varphi + \psi\sqrt{\eta}$:

$$-2\sigma = \varphi + \psi\sqrt{\eta}. \tag{32}$$

Then, by (31),

$$(\varphi + \psi\sqrt{\eta})^3 = 3(\varphi + \psi\sqrt{\eta}) - 1. \tag{33}$$

Just as we got (31) from (23), so from (33) we get

$$(-2\varphi)^3 = 3(-2\varphi) - 1,$$

i.e.,

$$(-2\varphi)^3 - 3(-2\varphi) + 1 = 0. \tag{34}$$

But, by (30), taking x as -2φ, we get

$$(-2\varphi)^3 - 3(-2\varphi) + 1 = (-2\varphi - \sigma - \tau\sqrt{\vartheta})(-2\varphi - \sigma + \tau\sqrt{\vartheta})(-2\varphi + 2\sigma).$$

Hence, by (34),

$$(-2\varphi - \sigma - \tau\sqrt{\vartheta})(-2\varphi - \sigma + \tau\sqrt{\vartheta})(-2\varphi + 2\sigma) = 0.$$

Hence at least one of the three factors must be 0. Thus either

$$-2\varphi = \sigma + \tau\sqrt{\vartheta} \quad \text{or} \quad -2\varphi = \sigma - \tau\sqrt{\vartheta} \quad \text{or} \quad \varphi = \sigma. \tag{35}$$

But this third case is impossible; for, if φ were equal to σ, (32) would yield

$$\sqrt{\eta} = -\frac{3\varphi}{\psi},$$

thus implying that $\sqrt{\eta}$ was expressible using no radicals of as high order as $\sqrt{\eta}$ except other radicals already occurring in -2σ—contrary to the original italicized description of ζ. So one of the remaining alternatives in (35) must hold:

$$-2\varphi = \sigma \pm \tau\sqrt{\vartheta}.$$

But then

$$\sqrt{\vartheta} = \pm\frac{2\varphi + \sigma}{\tau},$$

which would again be counter to the original description of ζ.

So the initial assumption that the angle DAB of FIGURE 5 could be trisected by ruler and compass has led to contradiction.

On the Symmetry Group of the Dodecahedron

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1. Introduction In this note we offer a proof of a well-known theorem on the group of symmetries of the dodecahedron, and thus of its dual, the icosahedron, which involves only the most easily visualized features of the solid, employing arguments from elementary group theory. This approach differs from that of the previous commonly-known proofs which entail visualization of (embedded or projected) supplementary figures associated with these solids (cf., [**1**, p. 47]).

We let P denote a dodecahedron with center O and let λ denote the (nonrealizable) symmetry taking each point T of P to the point symmetric to T in O. Pairs of opposite faces $\{F, \lambda(F)\}$ and a distinguished face F and vertex V are labeled as shown in FIGURES 1 and 2. (Visible faces have boldface numbers.) We let G denote the group of symmetries of P which are realizable (by motions) in three-space. In §2, using arguments from elementary group theory that do not involve Sylow theory, we prove:

THEOREM A. $G \cong A_5$ (*the alternating group on* 5 *letters*).

In §3, for completeness, we identify the full group G' of symmetries of P.

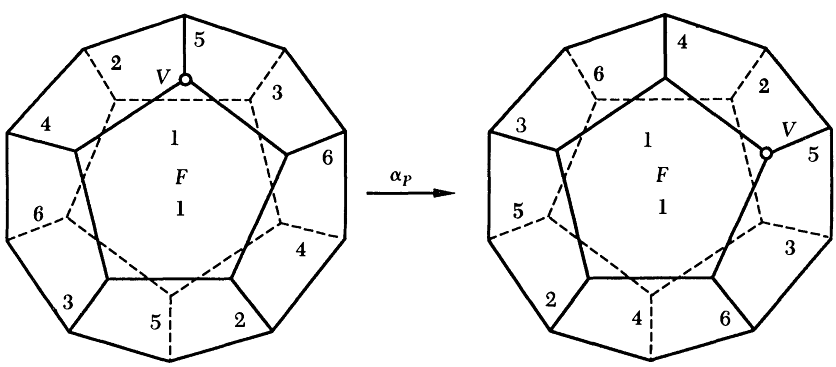

FIGURE 1

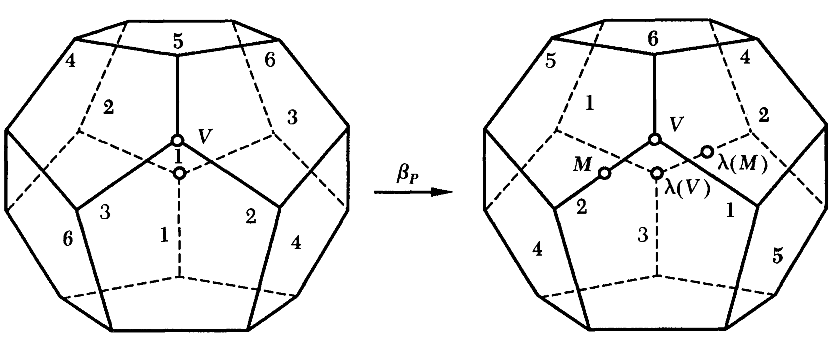

FIGURE 2

2. Proof of Theorem A We first observe that $|G| = 60$. For, given any face F' of the 12 faces of P and any vertex V' of the 5 vertices of F', it is clearly possible to move P around O in such a fashion that F and V occupy the former positions of F' and V'. (The positions of the remaining vertices of F, and hence of P are thus completely determined since orientation—clockwise or counterclockwise—is preserved on the surface of P.) Further, we note that each $\sigma_P \in G$ determines a permutation σ of the six pairs of faces of P, and that each such permutation determines at most one $\sigma_P \in G$, as the vertex V is surrounded by faces labeled 1, 2 and 3 in clockwise order, and V is on F. We conclude that G (a group under composition of mappings) is isomorphic to a subgroup H of $S = S_6$, the symmetric group on 6 letters. The rotation α_P depicted in FIGURE 1, about the axis joining the centers of F and $\lambda(F)$, corresponds to $\alpha = (1)(2,3,4,5,6) \in H$. Also, the rotation β_P about $V\lambda(V)$ depicted in FIGURE 2 corresponds to $\beta = (1,2,3)(4,6,5)$. We observe (multiplying permutations from right to left) that $\delta = \alpha\beta = (1,3)(2,4)(5)(6)$, $\omega = \beta\alpha\beta^{-1} = (1,6,4,5,3)(2)$, $\rho = \alpha^3\omega^2 = (1,2,5)(3,4,6)$ and $\gamma = \rho\delta\rho^{-1} = (1)(3)(2,4)(5,6)$ are in H. Since $\gamma\delta = \delta\gamma$, the group $\langle \gamma, \delta \rangle$ generated by δ and γ is a copy of the Klein four-group. Also, by Lagrange's theorem, $4 \mid |\langle \alpha, \beta \rangle|$. Since $5 = o(\alpha)$ and $3 = o(\beta)$, we have $15 \mid |\langle \alpha, \beta \rangle|$ also, so that $60 \mid |\langle \alpha, \beta \rangle|$ and $H = \langle \alpha, \beta \rangle \cong G$.

We will show that A_5 is also isomorphic to H. We begin by observing that A_5 contains 24 ($= 4!$) 5-cycles and thus 6 subgroups H_i' of order 5. For each $g \in A_5$ we define $(f[g])(i)$ to be j, where $gH_ig^{-1} = H_j$. Clearly $f[g]$ is one-to-one so that $f[g] \in S$. Since

$$(g_1g_2)H_i(g_1g_2)^{-1} = g_1(g_2H_ig_2^{-1})g_1^{-1}, (f[g_1g_2])(i) = f[g_1](f[g_2](i)) \quad \text{for all } i,$$

and thus the mapping $T\colon A_5 \to S$ given by $T(g) = f[g]$ for all $g \in A_5$ is a homomorphism.

We recall that to compute $\tau\mu\tau^{-1}$ one replaces every symbol in the disjoint cycle form of m by its image under τ. Thus, we can easily choose generators i for the subgroups H_i' of A_5 in such a manner that: $\sigma_1 = (1,2,3,4,5)$, $\sigma_2 = (1,2,3,5,4)$, $\sigma_1H_i'\sigma_1^{-1} = H_{i+1}'$ for $2 \le i \le 5$ and $\sigma_1H_6'\sigma_1^{-1} = H_2'$. (Since $\sigma_1\sigma_2\sigma_1^{-1} = (2,3,4,1,5) = (1,2,4,5,3)^3$ we take $\sigma_3 = (1,2,4,5,3)$. Similarly we choose $\sigma_4 = (1,2,5,4,3)$, $\sigma_5 = (1,2,5,3,4)$ and $\sigma_6 = (1,2,4,3,5)$.) Thus $f[\sigma_1] = (1)(2,3,4,5,6) = \alpha$. Letting $\tau_1 = (1)(4)(2,5,3)$, we find that $f[\tau_1] = (1,2,3)(4,6,5) = \beta$. Hence $T(A_5) = H$, and since $|A_5| = 60$, $A_5 \cong H$, so that $G \cong A_5$, as asserted. □

3. The full group G' of symmetries of P Arguing as in §2, we conclude that G' contains precisely 60 elements which reverse orientation, and these are just the mappings $\sigma\lambda$, where $\sigma \in G$. Since λ leaves the lines joining centers of opposite pairs of faces fixed, $\sigma\lambda = \lambda\sigma$ for all σ. Hence $G' = G \times \langle \lambda \rangle \cong A_5 \times C_2$ (where C_2 is a cyclic group of order 2) and G' has order 120. We note that G contains 24 rotations of order 5 about axes joining centers of opposite faces and 20 rotations of order 3 about axes joining opposite vertices. Also, it contains 15 rotations of measure π about axes joining midpoints $M, \lambda(M)$ of opposite edges (see FIGURE 2). Thus all 60 physically realizable symmetries of P are rotations, and conversely. (Of course, the geometers among us already knew this, as it is true for an arbitrary bounded solid.)

The author wishes to thank J. L. Brenner and Paul Guggenheim for their helpful suggestions.

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Walks Guided by the Sun

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1. Introduction The problem discussed in this note originates from a group of dedicated walkers. One day they decided to set neither the route to take, nor the finish of their walk, but to introduce a simple instruction to determine at each junction the direction to take: choose the path that is closest to the direction of the sun. The actual walk one makes under this instruction depends, of course, on the time of start and on the lay of the land. The general direction, however, changes from easterly in the morning, to southerly in midday, and westerly in the afternoon. An intriguing question for the walkers was: Is there a possibility that we get home again? or, less stringent: Shall we cross our earlier path? In this paper the trajectory of the walker is studied, albeit under simplifying conditions. The lay of the land is eliminated by asserting that there are no restrictions in the directions to go, just as for ships on high seas. Furthermore the walk shall not be restricted to the daytime with the sun above the horizon. At night the position of the sun is known, although invisible, and the direction to go is determined. The path of a walker (or a ship) under these conditions shall be referred to as "sunwalks." The shape of the sunwalk is the subject of this paper.

2. Position of the sun in the sky The position of the sun is characterized by the position, where the walker sees the sun on a unit sphere with him or her as centre. A cartesian left-handed coordinate system is used. The origin is chosen at the start of the walk. The X-, Y- and Z-axes are pointing towards South, West and Zenith, respectively.

Three angles are important. One is determined by the season at the time of the walk: the *season-angle* α. It is $0°$ at the equinox, March 21 and September 21. It varies between $-23.5°$ on December 21 to $+23.5°$ on June 21. The angle between the North Star and the sun is $90° - \alpha$. Since we shall neglect the small variation in α during the walk, the sun moves along a small circle on the unit sphere, with constant angular velocity Ω equal to $15°/\text{hour} = \pi/12$ rad/hour. Its actual position is determined by another relevant angle: the *hour-angle* ϕ, which increases from zero at noon, linearly with sun-time.

$$\phi = \Omega(t - 12). \tag{1}$$

The third important angle is the *latitude*, θ, of the location where the walk is taking place. The value of θ is positive on the northern hemisphere. It is considered constant during the entire walk.

In FIGURE 1 most of the characteristics introduced up till now are illustrated. In part a) a projection on the plane $Y = 0$ is given. The circle is the intersection with the unit-sphere. The sun-trajectory is a straight line. The angle α determines its distance to the origin and $\pi/2 - \theta$ the tilt with respect to the plane of the walk. In part b) the projection on the plane of the walk is shown. The path of the "sun" is now an ellipse. It is tangent to the unit-circle in the directions where the sun rises and sets. The numbers in the figure refer to the sun-time in hours, t.

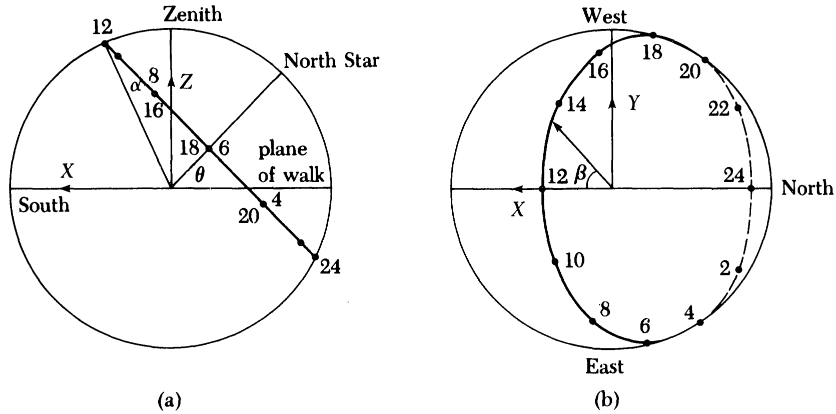

FIGURE 1
The sun is represented by a point on a unit-sphere with the walker as centre. The daily trajectory is a (small) circle. In part (a) the projection is shown on the plane through Zenith and South, in part (b) in the plane of the walk. The numerals give the sun-time in hours.

It is not difficult to show, that the position of the sun is given by

$$\begin{aligned} X &= \sin\theta\cos\alpha\cos\phi - \cos\theta\sin\alpha \\ Y &= \cos\alpha\sin\phi \\ Z &= \cos\theta\cos\alpha\cos\phi + \sin\theta\sin\alpha. \end{aligned} \tag{2}$$

The angle β, also indicated in FIGURE 1, represents the direction in which the walk has to proceed. It is measured from South, and positive towards West.

$$\cos\beta = X/\sqrt{1-Z^2}, \qquad \sin\beta = Y/\sqrt{1-Z^2}. \tag{3}$$

Its value remains undetermined for $|Z| = 1$, $X = Y = 0$: the sun in Zenith, or Nadir. However, no complication arises, since the sun remains there only a vanishingly short time. It may be noted that the motion of the sun through Zenith is always from East to West, and through Nadir from West to East.

3. Mathematical expression for the sunwalk Time enters in our problem in the angular velocity, Ω, of the sun and the velocity, v, of the walker. We shall eliminate time itself in favor of the hour-angle. All distances then can be measured in units of a standard-length, $S = v/\Omega$. The coordinates of the walker, measured in this unit, are given by x and y. With these definitions, the normalized coordinates for a sunwalk starting at an arbitrary hour-angle, are found to be:

$$x = \int \cos\beta\, d\phi; \qquad y = \int \sin\beta\, d\phi. \tag{4}$$

The lower and upper bounds of the integrals are equal to the values of ϕ at the start and the end of the walk.

At one position the sunwalk can be determined by simple reasoning, namely at the North Pole. There the height of the sun above the horizon is independent of the hour-angle: $Z = \sin\alpha$. Since the sun moves with constant angular velocity, the angle β increases linearly with time ($\beta = \phi$!). The constant velocity of the walker, now necessarily leads to a circle as sunwalk. The radius follows from the length of the path in 24 hours: $24 \cdot v/2\pi = v/\Omega = S$. In standard units the equation for the sunwalk at the North- (or South-) Pole now reads:

$$\theta = \pm 90°: x^2 + (y-1)^2 = 1; \text{ independent of } \alpha.$$

The motion of the sun along the sky shows mirror-symmetry with respect to the plane through Zenith and South. Equivalent positions occur for ϕ and $-\phi$. Such a change in ϕ leaves the magnitude of X, Y, Z and β unchanged, but the sign of Y and β reverses. However, $\cos\beta$ is again an even function of ϕ. Hence one must expect that the following holds:

$$x(\phi) - x(0) = x(0) - x(-\phi).$$

On the other hand, $\sin\beta$ is an odd function of ϕ and one obtains

$$y(\phi) - y(0) = y(-\phi) - y(0).$$

Apparently the line $x(0)$ is a mirror-line for the sunwalk. The same reasoning can be used for the two hour-angles $\pi + \phi$ and $\pi - \phi$. Again y and β change sign only. The conclusion must be that the line $x(\pi)$ is also a mirror-line for the sunwalk.

As far as the shape of the sunwalk is concerned, one needs to consider positive values for α and θ only. In TABLE I is shown which change in orientation takes place with a change in sign.

TABLE I

	characteristic angle for			time of		
season	latitude	sun-time	walk	day	year	orientation
α	θ	ϕ	β	t	y	
α	$-\theta$	$\pi+\phi$	$-\beta$	$t \pm 12$	y	mirror-image mirror-line: North-South
$-\alpha$	θ	$\pi+\phi$	$\pi+\beta$	$t \pm 12$	$y \pm \frac{1}{2}$	inversion
$-\alpha$	$-\theta$	ϕ	$\pi-\beta$	t	$y \pm \frac{1}{2}$	mirror-image mirror-line: East-West

The integrals for x and y (eq. 4) can be reduced to well-known forms by using the elevation of the sun, Z, as variable instead of the hour-angle ϕ. However, for a given value of Z the value of ϕ is not unique:

$$Z(-\phi) = Z(\phi) = Z(2\pi - \phi).$$

It is necessary to limit the interval in ϕ to the region 0 to π. The boundaries of the region are:

$$\phi = 0: Z = \cos(\theta - \alpha); \qquad \phi = \pi: Z = -\cos(\theta + \alpha).$$

Note that in this interval the value of y is nonnegative. The sunwalk for other hour-angles is found by (repetitive) use of the mirror-symmetry about the lines $x(0)$ and $x(\pi)$. Since

$$dZ = -Y\cos\theta\, d\phi,$$
$$X = (Z\sin\theta - \sin\alpha)/\cos\theta,$$
$$Y\cos\theta = \sqrt{\{\cos(\theta-\alpha) - Z\}\{\cos(\theta+\alpha) + Z\}},$$

the equations for x and y now read

$$x(\phi) - x(0) = -\tan\theta \int (Z - \sin\alpha/\sin\theta)\, dZ/\{Y\cos\theta\sqrt{1-Z^2}\};$$

$$y(\phi) - y(0) = -\int dZ/\{\cos\theta\sqrt{1-Z^2}\}.$$

The lower bound for the integrals is $\cos(\theta - \alpha)$, and the upper bound the value of Z that corresponds to the still-free-to-choose value of ϕ. The expression for x contains elliptical integrals in a noncanonical form. For $\alpha = 0$ and $|\theta| = |\alpha|$ the expression for x can be integrated directly; but the results do not contribute to a better understanding. They were used, however, to verify the accuracy in the numerical calculations, that were needed in all other cases.

The integral for y is

$$y(\phi) - y(0) = (\arccos(Z) - \theta + \alpha)/\cos\theta.$$

Note that $\arccos(Z)$ is equal to the angle between the sun and Zenith. For any Z there is only one value of $\arccos(Z)$ between 0 and π. After 24 hours the same value for $\arccos(Z)$ is reached and hence also the same value for y. In other words the East-West component of the sunwalk is periodic with a period of 24 hours.

4. Results and discussion Numerical calculations have been carried out to determine the shape of the sunwalks at other latitudes and for different seasons. A few examples will be shown in FIGURES 2–5. The season-angle is kept fixed at 20°, corresponding to a date of almost June 21. In the figures the upward direction is North and West is to the left. The numerals give the sun-time in hours. The dots give the position of the walker on the full hours.

In FIGURE 2 the latitude is 45° North. The motion of the sun for this situation was given in FIGURE 1. At noon the sun is in the South and the walker goes South, down in the figure. But in the course of the afternoon the direction changes to West. From FIGURE 1 it is clear that the sun is in the West not long after 16h. The sunwalk is in agreement: the tangent is pointing West shortly after 16h. At midnight the walker

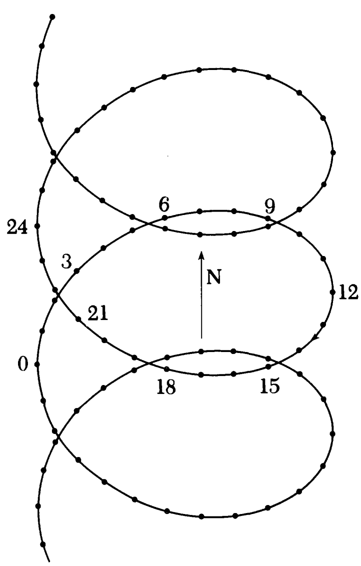

FIGURE 2
The sunwalk for latitude 45° North and season-angle 20°. The numerals give the sun-time in hours.

goes North. It must be noted that the change in direction from South to West took place in slightly more than 4 hours, whereas the change from West to North takes slightly less than 8 hours! Hence the path is less curved in the night than in the daytime. Shortly before 8h the sun is in the East and the walker goes East. At noon the same meridian is reached as at the start of the previous day at noon. But there is an overall North-South translation in the northern direction. This type of sunwalk is characteristic for those latitudes and seasons, for which $|\theta| > |\alpha|$.

Further North the daily translation is smaller, until at the Pole the daily translation is equal to 0. The sunwalk is then closed and a circle, as was shown above.

Going South the daily translation increases. In FIGURE 3 the latitude is 25° North. The double-point in the sunwalk now lies close to the position where the sun rises and sets ($Z = 0$; $\cos\phi = -\tan\theta\tan\alpha$). Apparently this combination of latitude and season is such that the daylight part of the sunwalk is practically closed. The walker starting at sunrise is home again at sunset.

Still further South the latitude $\theta = \alpha$ is reached where the sun goes through Zenith. This passage is always from East to West. So, shortly before noon the walker goes East, whereas immediately after noon the walker has to go West, the opposite direction! The sunwalk now apparently has a cusp. It is shown in FIGURE 4. Still further South the sun does not leave the northern half of the sky. The sunwalk now has a northern component for the full 24 hours. There are no longer double-points in the sunwalk. In FIGURE 5 the sunwalk for 15° North (season-angle still 20°) is given. The sunwalk on the equator shows 2-fold symmetry in the positions at 6h and 18h.

In FIGURE 6 the daily translation in North-South is given as a function of latitude for different season-angles. The following conclusions can be drawn. During the equinox ($\alpha = 0$) the North-South translation per day is zero, independent of the latitude. The sunwalk is then closed everywhere; the walker returns home after a 24 hour walk anywhere on earth. The translation per day grows with increasing season-angle. It is maximum at the equator, but remains zero at the pole. The points indicated by a dot refer to situations where the sun passes through Zenith and the

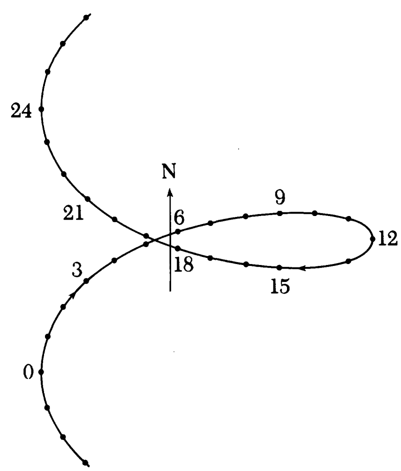

FIGURE 3
The sunwalk for latitude 25° North and season-angle 20°. The numerals give the sun-time in hours.

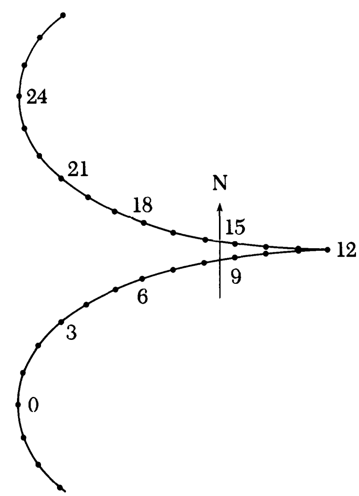

FIGURE 4
The sunwalk for latitude 20° North and season-angle 20°. The numerals give the sun-time in hours.

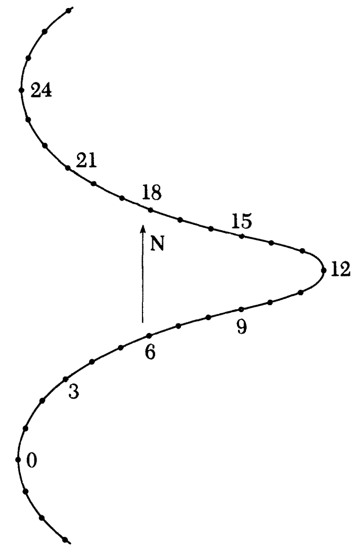

FIGURE 5
The sunwalk for latitude 15° North and season-angle 20°. The numerals give the sun-time in hours.

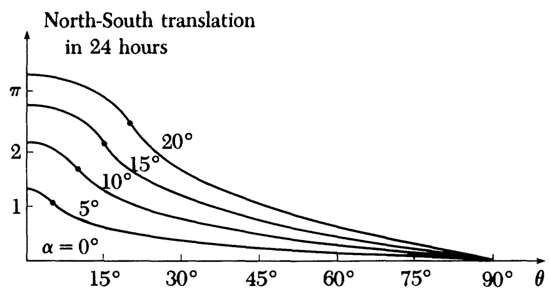

FIGURE 6
The North-South translation per day as a function of latitude for different season-angles.

sunwalk shows a cusp. It looks like there is an inflexion-point in the drawn lines at the dotted points. No essential differences are observed for season-angles larger than 23.5°, which do not occur on earth, however. For negative values of α and/or θ the sunwalk can be determined by using the symmetry given in TABLE I.

The East-West span of the sunwalk is the difference in y between the extremes at noon and at midnight. For the equinox ($\alpha = 0°$) the span decreases gradually from π at the equator, where the sunwalk is a straight line East-West, to 2 at the North Pole, where the sunwalk is the unit-circle. In general one obtains

$$y\text{-span} = (\pi - 2|\theta|)/\cos\theta; \qquad |\theta| \geq |\alpha|,$$
$$y\text{-span} = (\pi - 2|\alpha|)/\cos\theta; \qquad |\theta| \leq |\alpha|.$$

The question the walkers asked themselves: "Is there a possibility to return home?" can be answered affirmatively, provided $|\theta| > |\alpha|$, because then only a double-point is present in the sunwalk.

A Golden Frustum

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This investigation started when I was considering the remarkable fact that a cone inscribed in a cylinder uses up exactly one-third of the volume of the cylinder. How might we divide the remaining volume of the cylinder in half, so the partitioning into thirds is complete? As shown in FIGURE 1, the divider could be a frustum of a taller cone. The frustum would share its larger base with the cylinder and the cone. In the other base of the cylinder, where the vertex of the cone is, there would be a smaller copy of the base, reduced according to some shrink factor. When you learn what the shrink factor turned out to be, I hope you will share a little of my surprise and pleasure.

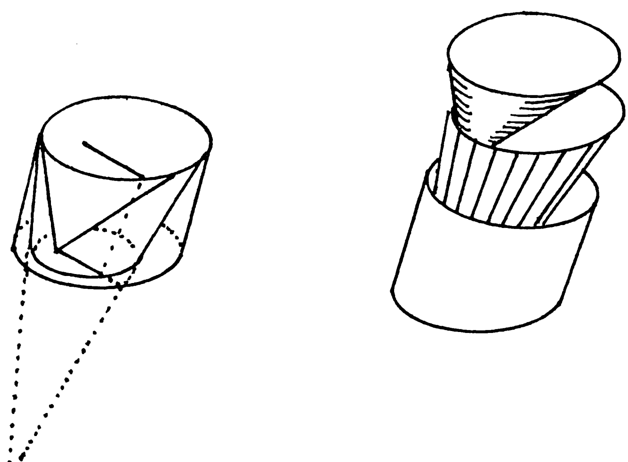

FIGURE 1

The result applies for any general cones inscribed in general cylinders, including, for instance, pyramids inscribed in prisms. The side elements of the cylinder need not be perpendicular to the bases.

The volume of a frustum can be expressed in terms of its altitude a, any basic length R in its base (a radius, diameter, perhaps an edge length), and the corresponding length r in the similar but smaller base. To derive the formula we picture the frustum as formed from a solid cone (the "full cone") by chopping off a "vertex cone" by a plane parallel to the base [FIGURE 2].

The area of the base is proportional to the square of the basic length R, say it is kR^2. The altitude of the full cone is the sum $h+a$ of the altitude of the vertex cone and the altitude of the frustum. The proportion $R/r=(h+a)/h$ yields the formulas $h=ar/(R-r)$ and $h+a=aR/(R-r)$, so the volume of the frustum is

$$kR^2(h+a)/3-kr^2h/3=ka(R^3-r^3)/3(R-r), \quad \text{or } ka(R^2+Rr+r^2)/3.$$

Our trisection condition is that this volume be 2/3 that of the cylinder:

$$ka(R^2+Rr+r^2)/3=(2/3)kR^2a, \quad \text{from which } R^2-Rr-r^2=0.$$

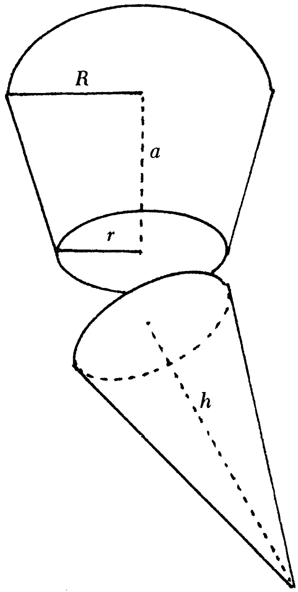

FIGURE 2

Division of this equation by r^2 shows that the ratio of R to r satisfies the quadratic equation $x^2 - x - 1 = 0$. The positive root of this equation is the famous "golden section" $G = (1 + \sqrt{5})/2$. The shrink factor $1/G$ reduces the linear dimensions of the base of the cylinder to the corresponding linear dimensions of the smaller base, justifying the term "golden" frustum.

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Five Mathematical Clerihews

Alan Turing
Found alluring
Machines whose only fault
Was that they would not halt.

There were so many kinsmen Bernoulli
That keeping them straight would unduly
Have tired and worn to a frazzle
The record-keepers of Basel.

A hobby of David Hilbert's
Was to raise and eat square filberts.
Down the tubes
Went his cubes.

Carl Friedrich Gauss
Brought down Euclid's house,
Which would have pleased nary
Lambert nor Saccheri.

The quintic didn't awe
Evariste Galois.
The roots permuted:
His groups weren't suited.

—Karl David
Wells College
Aurora, NY 13026

Configurations Arising from the Three Circle Theorem

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Problems involving tangent circles have been investigated in both the West and the East. In Japan such problems were of special interest in the Wasan period. ("Wasan" refers to Japanese mathematics developed independently of Western science in the 17th–19th centuries.) Wasan researchers often wrote their problems and solutions on a framed board, which was dedicated to a shrine or a temple to express gratitude to the gods. Most such problems were geometric, and the figures were beautifully drawn in color. The board is called a **sangaku** (votive tablet on mathematics).* It was also a means to publish a discovery or to propose a problem. So Wasan mathematicians thereby left many geometric results (in particular on tangent circles or spheres). We will refer to two such results (for extensive references see [**5**] or [**7**]).

The author discovered an interesting configuration and consequence of this theorem while considering a problem of the Wasan mathematician Ushijima [**8**]: "In FIGURE 1, C_1, C_2, C_3, C_4 are in circles of the triangles ABD, ADC, $AD'C'$, $AB'D'$ respectively. Given the diameters of C_2, C_3, C_4, find the diameter of C_1." The solution essentially states that

$$\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4}, \tag{1}$$

where r_i is the radius of C_i. We shall prove this equation and investigate the symmetry suggested by this.

For two circles, their external center of similitude is the point which divides the line between the centers externally in the ratio of the radii and also is the intersection of external common tangents if they can be drawn. Similarly, the internal center of similitude can be defined. We will use the Three Circle Theorem (this can be easily proved by the theorem of Menelaus [**3**, p. 137] and also see [**2**], [**4**] and [**6**]): In the Euclidean plane, given three circles with distinct radii and noncollinear centers, the six centers of similitude of pairs of circles lie in threes on four straight lines (FIGURE 2).

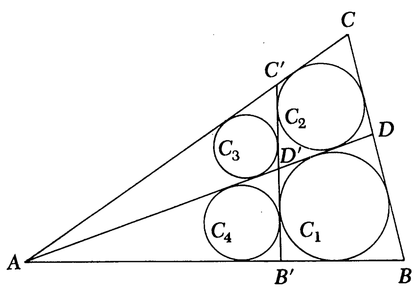

FIGURE 1

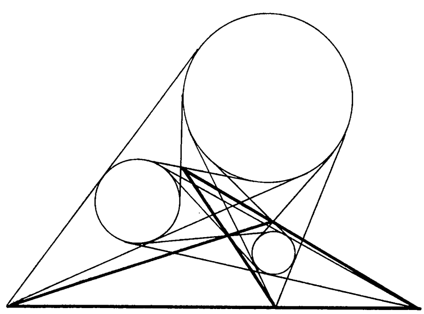

FIGURE 2

*"Sangaku" does not mean the votive tablet in our context but mathematics itself.

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We first embed the Euclidean plane in a projective plane by adding the line at infinity. If we now consider the Three Circle Theorem, we can allow circles with equal radii (FIGURE 3).

We consider cycles (oriented circles) and rays (oriented lines) instead of circles and lines. A cycle and a ray touch each other if their orientations at the point of tangency are the same. For two cycles with the same orientation, the center of similitude is the external center of similitude when we regard them as two circles. If two cycles have the opposite orientations, their center of similitude is the internal center of similitude of the two circles. Therefore, the center of similitude of two cycles is the intersection of the common rays touching them if they exist. For a cycle C and a ray x, $\bar{C}$ and $\bar{x}$ denote the cycle and the ray along C and x having the opposite orientations to C and x respectively. For oriented circles the Three Circle Theorem is stated as follows.

(2) *Given three cycles, the three centers of similitude lie on one line (in the projective plane containing the Euclidean plane,* FIGURE 4).

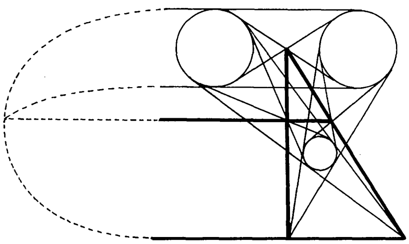

FIGURE 3

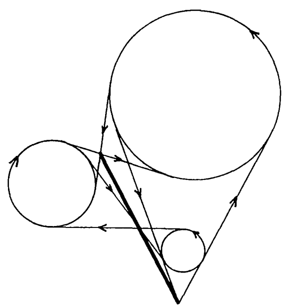

FIGURE 4

Let P_{ij} denote the center of similitude of cycles C_i and C_j. If C_i and C_j touch a ray x or $\bar{x}$, P_{ij} lies on x. The following lemma explains the symmetry of FIGURE 1.

(3) LEMMA. *Let x, y, $\bar{x}$ be distinct rays and C_1 touches $x, \bar{y}$; C_2 touches x, y; C_3 touches $\bar{x}, y$; C_4 touches $\bar{x}, \bar{y}$. Then $P_{12} = P_{34}$ if and only if $P_{23} = P_{41}$ (in the projective plane,* FIGURE 5).

Proof. As noted above, P_{12}, P_{34} lie on x and P_{41}, P_{23} lie on y. Suppose $P_{12} = P_{34}$. Let u be the line (see (2)) through P_{12}, P_{23}, P_{31}; then P_{23} is the intersection of u and y. Since $P_{12} = P_{34}$ and P_{31} lies on u, P_{41} lies on u. Therefore, $P_{23} = P_{41}$. Similarly $P_{23} = P_{41}$ implies $P_{12} = P_{34}$.

We construct our configuration with the aid of this lemma (see FIGURE 8).

(4) Construction. In the following, x_i and y_i indicate rays passing through two arbitrary points X and Y, respectively, with $x_i \neq \bar{x}_{i+1}, y_i \neq \bar{y}_{j+1}$, and (i, j) denotes a cycle for any integer i and j.

Draw $(0,0)$ such that $(0,0)$ does not intersect the line through X and Y. Draw the four rays x_0, y_0, x_1, y_1 such that $x_0, y_0, \bar{x}_1, \bar{y}_1$ touch $(0,0)$. Then construct $(1,0), (2,0), x_2, x_3$ such that $(1,0)$ touches $x_1, \bar{x}_2, y_0, \bar{y}_1$ and $(2,0)$ touches $x_2, \bar{x}_3, y_0, \bar{y}_1$. Similarly for any integers m, n we get $(m,0), (0,n), x_m, y_n$ such that $(m,0)$ touches $x_m, \bar{x}_{m+1}, y_0, \bar{y}_1$, and $(0,n)$ touches $x_0, \bar{x}_1, y_n, \bar{y}_{n+1}$. Then by the lemma we can draw $(1,1), (1,2), (2,1)$ such that $(1,1)$ touches $x_1, \bar{x}_2, y_1, \bar{y}_2$; $(1,2)$ touches $x_1, \bar{x}_2, y_2, \bar{y}_3$; $(2,1)$ touches $x_2, \bar{x}_3, y_1, \bar{y}_2$. Similarly we get a cycle (m,n) touching $x_m, \bar{x}_{m+1}, y_n, \bar{y}_{n+1}$ for any integer m and n.

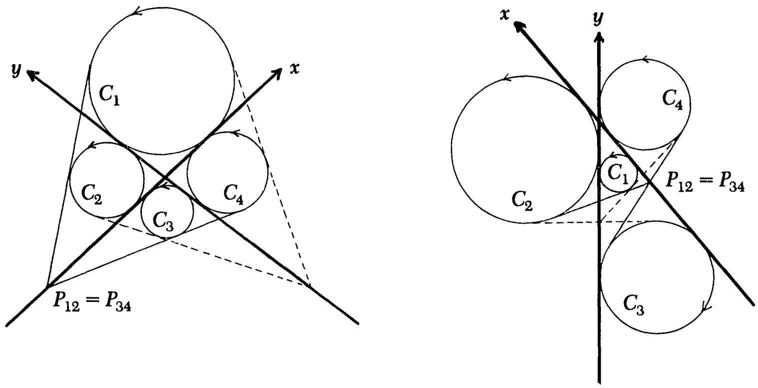

FIGURE 5

X and Y will be called the vanishing points of this configuration. In FIGURE 6 both vanishing points lie on the line at infinity, in FIGURE 7 just one. FIGURE 8 shows a configuration in which both of the vanishing points are ordinary.

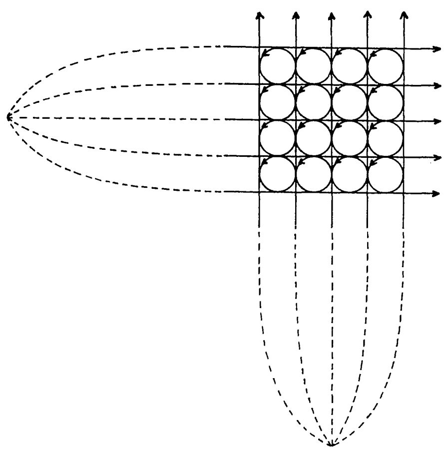

FIGURE 6

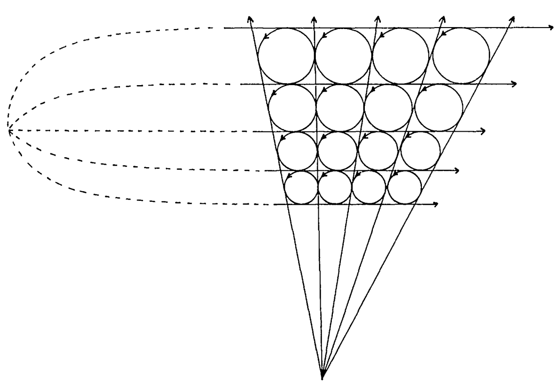

FIGURE 7

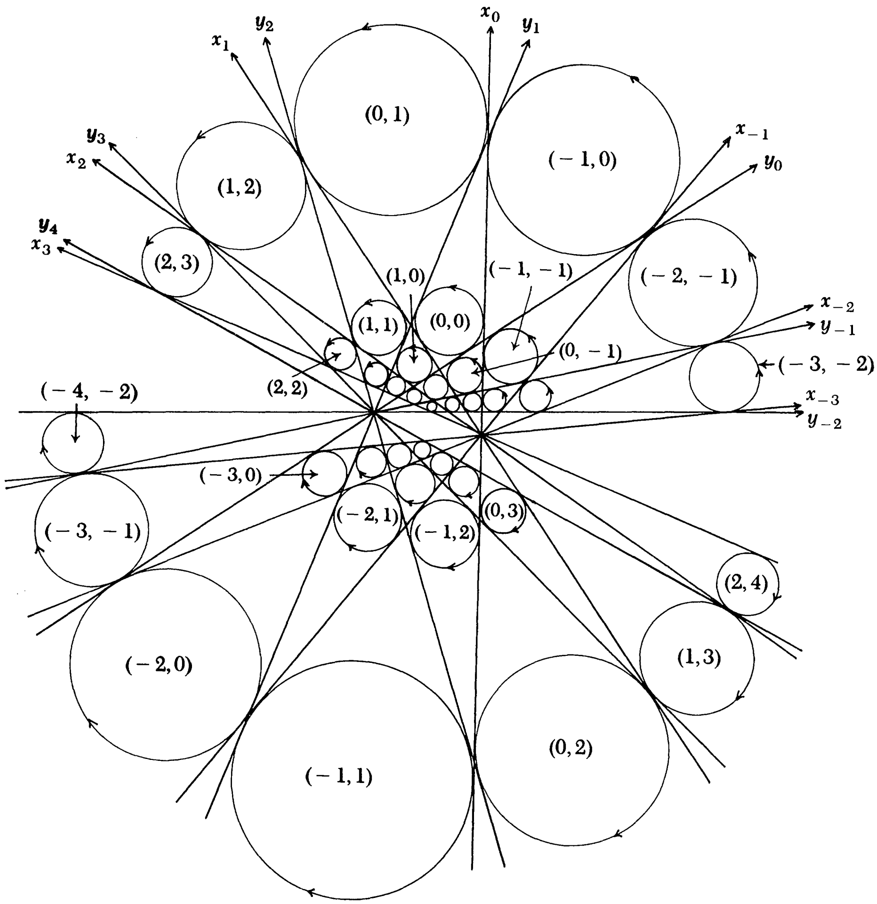

FIGURE 8

In this construction it may happen that at some stage, x_i and y_0 are parallel and $(i, 0)$ does not exist. However we can introduce a cycle at infinity touching parallel rays. For two rays x and y, where both x, y and $x, \bar{y}$ are non-parallel, we postulate the existence of a cycle C_1 touching x, y and all the rays parallel to x or y at infinity. Then for any ordinary cycle C_2, P_{12} can be defined as the intersection of the common tangent rays. If C_1 and C_2 are cycles at infinity, we choose an arbitrary ordinary cycle C_3 and define P_{12} as the intersection of the line through P_{13}, P_{23} and the line at infinity. For such cycles, we define their curvature (reciprocal of the radii) to be 0.

We conclude with some properties of this configuration of cycles and rays.

In FIGURE 6, since any two cycles have the same radii, their center of similitude lies on the line at infinity; that is, for any two cycles the center of similitude lies on the line through the two vanishing points. Every configuration constructed above has this property. Choose any two cycles (i, j) and (m, n). We consider the three cycles $(i, j), (m, n), (m, j)$. Since the center of similitude of (m, n) and (m, j) is X and the center of similitude of (i, j) and (m, j) is Y, the center of similitude of (i, j) and (m, n) lies on the line through X and Y by (2). Therefore:

(5) THEOREM. *For any two cycles in the configuration, the center of similitude lies on the line through the two vanishing points.*

For a cycle C_i, let r_i denote its radius. We consider the sign of the radius of a cycle to be plus if its orientation is anti-clockwise, otherwise minus. In FIGURE 9, u touches $\bar{C}_1$ and C_4. Let h be the distance between P_{14} and u, where we define the sign of h to be plus if P_{14} lies on the right side of u, and minus if on the left side of u. Then by similar triangles

$$\frac{h}{r_1}=\frac{h+2r_4}{r_4}\ \text{(FIGURE 9a)} \quad \text{or} \quad \frac{r_1-h}{r_1}=\frac{h+r_4}{-r_4}\ \text{(FIGURE 9b)},$$

and, in any case,

$$\frac{1}{r_1}-\frac{1}{r_4}=\frac{2}{h}.$$

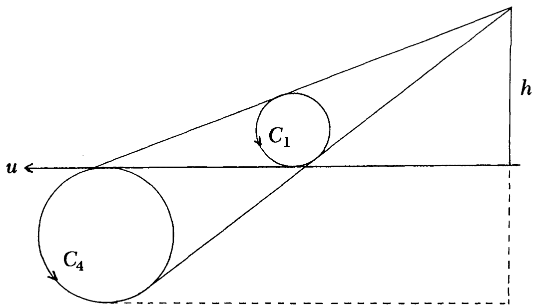

FIGURE 9a

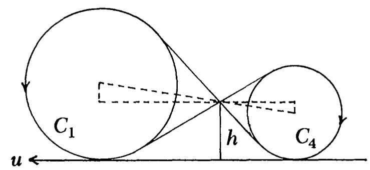

FIGURE 9b

Therefore, if the two centers of similitude of C_1, C_4 and C_2, C_3 coincide, and a ray u touches $\bar{C}_1, \bar{C}_2, C_3, C_4$, then we have

$$\frac{1}{r_1}-\frac{1}{r_4}=\frac{1}{r_2}-\frac{1}{r_3} \quad \text{or} \quad \frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4}.$$

This proves (1). In the configuration constructed in (4) let $[i, j]$ denote the curvature (the reciprocal of the radius) of cycle (i, j), then this result shows

$$[i,j]+[i+1,j+1]=[i+1,j]+[i,j+1]. \tag{6}$$

In FIGURE 10, u_1, u_2, u_3, u_4 touch C_0. C_1 touches $u_2, \bar{u}_3, u_4$; C_2 touches $u_3, \bar{u}_4, u_1$; etc. Then $r_1r_3=r_2r_4$. Indeed, calculating the distance between the two points of intersection $u_2\cdot u_3$ and $u_3\cdot u_4$, we have

$$r_0\left(\tan\frac{\{2,3\}}{2}+\tan\frac{\{3,4\}}{2}\right)=r_1\left(\cot\frac{\{2,3\}}{2}+\cot\frac{\{3,4\}}{2}\right),$$

where $\{i, j\}$ is the oriented angle measured from u_i to u_j. Hence,

$$r_1=r_0\tan\frac{\{2,3\}}{2}\tan\frac{\{3,4\}}{2}.$$

With similar formulas for r_2, r_3, r_4, we obtain $r_1r_3=r_2r_4$. (For FIGURE 10a the result itself appeared in [1].) By suitable rearrangement of the orientations in these figures

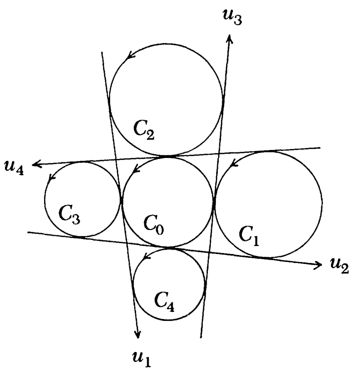

FIGURE 10a

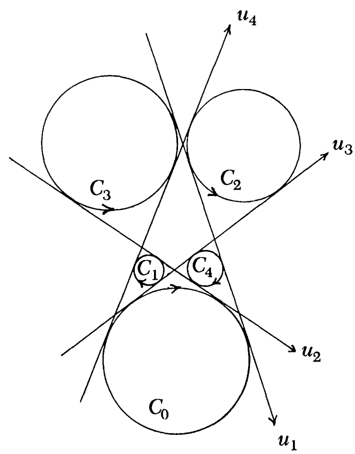

FIGURE 10b

(reverse the orientations of u_1 and u_2 for example), we have

$$[i, j][i+2, j] = [i+1, j-1][i+1, j+1]. \tag{7}$$

From (6) and (7) we may prove a concluding result by induction on m, n:

(8) THEOREM. *In the configuration, for any integers i, j, m, n, we have*

$$[i, j] + [i+m, j+n] = [i+m, j] + [i, j+n],$$

$$[i, j][i+m+n, j-m+n] = [i+m, j-m][i+n, j+n].$$

From this note we omit the case where $(0,0)$ intersects the line through the vanishing points, but in this case we can also construct this configuration except some special cases. We hope to discuss this in a later paper.

The author would like to thank the referees and the editor for their helpful suggestions in preparing this paper.

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The Gaps Between Consecutive Binomial Coefficients

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Among all combinatorial quantities, the binomial coefficients are unique. They are simple in concept and derivation, yet they appear in almost all the combinatorial identities and seem to give rise to an endless variety of problems. In this note we investigate one such problem which we believe to be new. Put in a simple way, the problem is to determine the smallest and largest "gaps" between consecutive binomial coefficients in any row of Pascal's triangle. More precisely, for positive integers n and k with $n \geqq 2$ and $1 \leqq k \leqq n$, we define

$$d_k(n) = \left| \binom{n}{k} - \binom{n}{k-1} \right|$$

and we let $d_m(n) = \min_{1 \leqq k \leqq n} d_k(n)$, and $d_M(n) = \max_{1 \leqq k \leqq n} d_k(n)$. Our purpose is to solve the following problems:

P1. Determine $d_m(n)$ and all k such that $d_k(n) = d_m(n)$.

P2. Determine $d_M(n)$ and all k such that $d_k(n) = d_M(n)$.

We start out with P1, which is the easier one. Due to symmetry, it clearly suffices to consider $d_k(n)$ for $1 \leqq k \leqq [(n+1)/2]$. Since

$$\binom{n}{k} - \binom{n}{k-1} = \frac{n!(n-2k+1)}{k!(n-k+1)!} = 0$$

if and only if $n - 2k + 1 = 0$, we see that $d_m(n) = 0$ if and only if n is odd and $d_k(n) = 0$ if and only if $k = (n+1)/2$. When n is even we shall show that $d_m(n) = n - 1$ except when $n = 4$ in which case direct inspection reveals that $d_m(4) = 2$. In fact, we shall show that in general if we define $d'_m(n) = \min\{d_k(n) | 1 \leqq k \leqq n,\ k \neq (n+1)/2\}$ (so that the aforementioned trivial fact that $d_m(n) = 0$ when n is odd is excluded), then $d'_m(n) = n - 1$ holds for all $n \neq 4$.

THEOREM 1. *For all $n \geqq 2$, except when $n = 4$, $d'_m(n) = n - 1$. Furthermore, if $n \neq 6$, then $d_k(n) = n - 1$ if and only if $k = 1$ or n, and if $n = 6$, then $d_k(6) = 5$ if and only if $k = 1, 3, 4,$ or 6.*

Proof. Since the cases when $n \leqq 6$ can be verified directly by inspection, we assume that $n > 6$ and use induction. Examining the appropriate rows in Pascal's triangle reveals that the theorem holds for $n = 7$ and 8. Due to symmetry and the fact that $d_1(n) = d_n(n) = n - 1$ we need only to show that $d_k(n) > n - 1$ for all k such that $2 \leqq k \leqq [n/2]$ where $n \geqq 9$. By Pascal's identity, we have:

$$\binom{n}{k} - \binom{n}{k-1} = \left\{ \binom{n-1}{k} + \binom{n-1}{k-1} \right\} - \left\{ \binom{n-1}{k-1} + \binom{n-1}{k-2} \right\}$$

$$= \left\{ \binom{n-1}{k} - \binom{n-1}{k-1} \right\} + \left\{ \binom{n-1}{k-1} - \binom{n-1}{k-2} \right\}. \tag{1}$$

If n is odd, then by the induction hypothesis, we have

$$\binom{n-1}{k} - \binom{n-1}{k-1} \geqq n-2 \quad \text{and} \quad \binom{n-1}{k-1} - \binom{n-1}{k-2} \geqq n-2.$$

Hence from (1) we obtain

$$\binom{n}{k} - \binom{n}{k-1} \geqq 2(n-2) = (n-1) + (n-3) > n-1. \tag{2}$$

If n is even then the above argument still holds except when $k = [n/2]$ in which case

$$\binom{n-1}{k} - \binom{n-1}{k-1} = 0.$$

By a second application of Pascal's identity we obtain from (1)

$$\binom{n}{k} - \binom{n}{k-1} = \binom{n-1}{k-1} - \binom{n-1}{k-2}$$
$$= \left\{ \binom{n-2}{k-1} - \binom{n-2}{k-2} \right\} + \left\{ \binom{n-2}{k-2} - \binom{n-2}{k-3} \right\}. \tag{3}$$

Since $n-2$ is even we have, by the induction hypothesis,

$$\binom{n-2}{k-1} - \binom{n-2}{k-2} \geqq n-3 \quad \text{and} \quad \binom{n-2}{k-2} - \binom{n-2}{k-3} \geqq n-3.$$

Hence from (3), we obtain

$$\binom{n}{k} - \binom{n}{k-1} \geqq 2(n-3) = (n-1) + (n-5) > n-1. \tag{4}$$

By (2) and (4) our proof is complete.

Now we turn to P2 the answer of which is somewhat less obvious than that of P1. The complete answer to this problem is contained in the next

THEOREM 2. *Let* $\tau = (1/2)(n+2-\sqrt{n+2})$. *Then*

$$d_M(n) = \binom{n}{[\tau]} - \binom{n}{[\tau]-1}.$$

Furthermore, $d_k(n) = d_M(n)$ *if and only if*

$$k = \begin{cases} [\tau] \text{ or } n-[\tau]+1 \text{ if } \tau \notin \mathbb{Z} \\ \tau-1, \tau, n-\tau+1 \text{ or } n-\tau+2 \text{ if } \tau \in \mathbb{Z} \end{cases}$$

where $\mathbb{Z}$ *denotes the set of integers.*

Proof. Due to symmetry we may again assume that $k \leqq [n/2]$. Direct computations show that

$$\left\{ \binom{n}{k} - \binom{n}{k-1} \right\} - \left\{ \binom{n}{k-1} - \binom{n}{k-2} \right\} = \frac{n!\sigma}{k!(n-k+2)!}$$

where

$$\begin{aligned}\sigma &= (n-k+2)(n-2k+1)-k(n-2k+3)\\ &= (n-2k+1)(n-2k+2)-2k\\ &= (2k)^2-(2n+4)(2k)+(n+1)(n+2)\\ &= \{2k-(n+2)\}^2-(n+2)\\ &= \{2k-(n+2)+\sqrt{n+2}\}\{2k-(n+2)-\sqrt{n+2}\}\\ &= 4(k-\tau)\{k-\tfrac{1}{2}(n+2+\sqrt{n+2})\},\end{aligned}$$

where $\tau=(1/2)(n+2-\sqrt{n+2})$. Since $k \leqq [n/2] < (1/2)(n+2+\sqrt{n+2})$ we conclude that

$$\binom{n}{k}-\binom{n}{k-1} \geqq \binom{n}{k-1}-\binom{n}{k-2}$$

if and only if $k \leqq \tau$ with equality holding if and only if $\tau \in \mathbb{Z}$. The statement of our theorem now follows immediately.

To illustrate Theorem 2, we consider an example.

Example. When $n=13$, $\tau=\frac{1}{2}(15-\sqrt{15}) \notin \mathbb{Z}$ and $[\tau]=5$. Thus the largest gap in the 13th row of Pascal's triangle occurs twice: once between $\binom{13}{4}$ and $\binom{13}{5}$, and the other between $\binom{13}{8}$ and $\binom{13}{9}$. These gaps have absolute value $\binom{13}{5}-\binom{13}{4}=1287-715=572$. When $n=14$, $\tau=6 \in \mathbb{Z}$. Thus the largest gaps in the 14th row of Pascal's triangle occur four times: between $\binom{14}{4}$ and $\binom{14}{5}$; $\binom{14}{5}$ and $\binom{14}{6}$; $\binom{14}{8}$ and $\binom{14}{9}$; and between $\binom{14}{9}$ and $\binom{14}{10}$. The absolute value of these gaps is $\binom{14}{5}-\binom{14}{4}=2002-1001=1001$.

Theorem 2 also indicates an interesting fact regarding consecutive binomial coefficients which form an arithmetic progression. It is well known [1, p. 54] that no four consecutive binomial coefficients can form an arithmetic progression. Our result in Theorem 2 implies that there are infinitely many triples of consecutive binomial coefficients which form an arithmetic progression. In fact, we have:

COROLLARY. *If $n>2$ such that $n+2$ is a perfect square, then $\binom{n}{\tau-2}$, $\binom{n}{\tau-1}$, and $\binom{n}{\tau}$ form an arithmetic progression where $\tau=(1/2)(n+2-\sqrt{n+2})$. Furthermore, the common difference of this arithmetic progression yields the largest gap in the nth row of Pascal's triangle.*

In closing, we point out that if, in the left half of each row in Pascal's triangle, we use a dot to represent the larger one of the binomial coefficients whenever a largest gap occurs and connect the dots in consecutive rows with line segments, then we would obtain a chain which has some kind of zigzag pattern and which has a "diamond" whenever $n>2$ is such that $n+2$ is a perfect square. We leave it to the readers to do the actual drawing.

Acknowledgement. This article was written while the first author was visiting Wilfrid Laurier University, July 1987–April 1988. The hospitality of WLU is greatly appreciated. This research was supported by the Natural Sciences and Engineering Research Council of Canada under grant A9121.

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PROBLEMS

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St. Olaf College

Proposals

To be considered for publication, solutions should be received by September 1, 1990.

1343. *Proposed by Ronald L. Graham, AT&T Bell Labs, Murray Hill, New Jersey.*

a. What is the behavior of the recursive sequence defined by $x_{n+2} = (1 + x_{n+1})/x_n$, with x_0, x_1 arbitrary positive numbers?

b. Answer the same question for $x_{n+3} = (1 + x_{n+1} + x_{n+2})/x_n$.

c. *Generalize (a) and (b).

1344. *Proposed by Keith Loseke, student, Buhler High School, Buhler, Kansas.*

Prove the following generalization of the formula $\gcd(a, b) = ab/\mathrm{lcm}(a, b)$:

$$\gcd(a_1, a_2, \dots, a_k) = \frac{P_0}{P_E},$$

where $a_1, a_2, \dots, a_k$ are positive integers, "gcd" and "lcm" are abbreviations for greatest common divisor and least common multiple, P_0 is the product of the lcm's of all subsets of $\{a_1, a_2, \dots, a_k\}$ with an odd number of elements and P_E is the similar product of the non-empty subsets of even numbers of elements.

1345. *Proposed by the Fullerton Problem Group, California State University, Fullerton, California.*

Let $A(\varepsilon)$ be the union of all maximal bounded closed intervals of $(0,1)$ such that $-\varepsilon \leqslant \sin(1/x) \leqslant \varepsilon$ for every $x \in A(\varepsilon)$. Denote by $S(\varepsilon)$ the sum of the lengths of all these intervals. Prove that

$$\lim_{\varepsilon \to 0} \frac{S(\varepsilon)}{\varepsilon} = \frac{1}{3}.$$

ASSISTANT EDITORS: CLIFTON CORZAT and THEODORE VESSEY, *St. Olaf College. We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics.. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren Larson, Department of Mathematics, St. Olaf College, Northfield, MN 55057.

1346. *Proposed by Jerry M. Metzger, University of North Dakota, Grand Forks, North Dakota.*

Consider a solitaire game played as follows. From an ordinary deck of 52 cards, lay out the top 8 cards face up in a row. If two or more cards have the same rank (for example, two eights or two jacks), pick two of them and cover them by two cards dealt face up from the top of the deck. Continue in this way as long as at least two of the cards show the same rank. The game ends if one cannot play, and is a win if the deck is exhausted. Prove that the game is a win working from the top of the deck to the bottom if and only if it is a win with the deck turned over (that is, working from the bottom to the top).

1347. *Proposed by James O. Chilaka, Long Island University, New York.*

Let $G = GL(n,\mathbf{R})$ be the group under multiplication of $n \times n$ matrices with real entries and non-zero determinant. Find the least number k of elements $g_1, g_2, \dots, g_k$ in G such that if a is in G and $a \cdot g_i = g_i \cdot a$, $i = 1,2,\dots,n$, then $a \cdot g = g \cdot a$ for all g in G.

CORRECTION: In problem 1342 in the February 1990 issue, the lower number in the binomial coefficient should be $j - i$, not $j - 1$.

Quickies

Answers to the Quickies are on page 133.

Q760. *Proposed by W. Weston Meyer, General Motors Research Laboratories, Warren, Michigan.*

Show that a polynomial with integer coefficients can have no integer zero that is larger, in absolute value, than every one of the coefficients.

Q761. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Canada.*

If $a_1, a_2, \dots, a_{n+1} > 0$, prove that

$$a_1 a_2 \dots a_{n+1}(a_1^{-n} + a_2^{-n} + \cdots + a_{n+1}^{-n}) \geqslant a_1 + a_2 \cdots + a_{n+1}.$$

Q762. *Proposed by Florentin Smarandache, Istanbul, Turkey.*

Prove that there exist an infinite number of primes which contain given digits, $a_1, a_2, \dots, a_m$, in the positions $i_1, i_2, \dots, i_m$, with $i_1 > i_2 > \cdots > i_m > 0$. (The "$i$th position" is the 10^ith digit.)

Solutions

An Involution on Sets of Subsets **April 1989**

1317. *Proposed by Allen J. Schwenk, Western Michigan University, Kalamazoo.*

Let $X_n = \{1,2,\dots,n\}$ and let S be any nonempty collection of subsets of X_n. Define S' to be the collection of all subsets of X_n that are subsets of an odd number of elements of S. For example, when $X_3 = \{1,2,3\}$ and $S = \{\{3\},\{1,2\},\{1,3\}\}$, we find $S' = \{\varnothing,\{2\},\{1,2\},\{1,3\}\}$. In Problem **1267**, Ronald Graham asked us to prove that $(S')' = S$.

Observe that occasionally $S' = S$, for example when $S = \{\varnothing,\{2\},\{1,2\},\{1,3\},\{2,3\}\}$. Characterize and count the collection of subsets for which $S' = S$.

Solution by Irl C. Bivens, Davidson College, Davidson, North Carolina.

We will show that there are $2^{2^{n-1}} - 1$ such collections and that any nonempty collection S satisfies $S' = S$ if and only if there exists a collection T such that S is the symmetric difference of T and T'.

Let V denote the Boolean ring consisting of the power set of the power set of X_n with addition given by the operation of symmetric difference and multiplication given by the operation of set intersection. The multiplicative identity element of V is given by the power set of X_n and it generates a subring of V isomorphic to the two element field Z_2. We may therefore regard V as a vector space over Z_2, with basis given by the singleton collections, and (hence) with dimension 2^n.

Define a mapping P: $V \to V$ by setting $P(S) = S'$ if S is nonempty and by letting $P(\varnothing) = \varnothing$. It is straightforward to show that P is a Z_2 linear mapping and it follows from Problem **1267** that $P^2 = I$. Therefore, 1 (in Z_2) is the only eigenvalue of P and the present problem asks us to characterize and count the corresponding collection of eigenvectors.

Note that if T is any element of V then $P(T + P(T)) = P(T) + P^2(T) = P(T) + T = T + P(T)$ so that the range of $I + P$ is a subspace of the space of eigenvectors of P. On the other hand, this space of eigenvectors is equal to the kernel of $P - I$ which (since we are over Z_2) is equal to the kernel of $I + P$. Thus, the range of $I + P$ is a subspace of the kernel of $I + P$ so that

$$\dim(\ker(I + P)) \geqslant \dim(\text{range}(I + P)).$$

In addition, the rank–nullity theorem implies that

$$\dim(\ker(I + P)) + \dim(\text{range}(I + P)) = \dim(V) = 2^n.$$

Therefore, to prove that $\ker(I + P) = \text{range}(I + P)$ it suffices to show that $\dim(\text{range}(I + P)) \geqslant 2^{n-1}$. Given any subset W of $\{2, 3, \ldots, n\}$, the set $E(W) = (I + P)(\{\{1\} \cup W\})$ is equal to the collection of *proper* subsets of $\{1\} \cup W$. In particular, W is an element of $E(W)$, so that as W ranges over the 2^{n-1} subsets of $\{2, 3, \ldots, n\}$, $E(W)$ varies over 2^{n-1} *distinct* elements of the range of $I + P$.

We claim that the 2^{n-1} element set, $Y = \{E(W)$: W is a subset of $\{2, 3, \ldots, n\}\}$, is linearly independent. To see this, note that in any linear combination of the vectors in Y, if W is a subset of $\{2, 3, \ldots, n\}$ of maximum order such that the coefficient of $E(W)$ is nonzero, then W is also an element of this linear combination. Hence, this linear combination is nonempty (i.e., nonzero) and Y is an independent set of vectors.

Consequently, $\dim(\text{range}(I + P)) \geqslant 2^{n-1}$ from which it follows that the space of eigenvectors of P is equal to the range of $I + P$, and is of dimension 2^{n-1}. Since we are over the field Z_2 there are $2^{2^{n-1}} - 1$ nonzero (nonempty) such eigenvectors.

Finally, for an "efficient" way of computing these eigenvectors, consider any nonempty collection T of subsets of X_n such that each subset in T contains 1 as an element. The analysis above shows that the mapping $T \to T + T'$ yields a one-to-one correspondence between such collections T and the collections S satisfying $S' = S$.

Also solved by Curtis Cooper, James Propp, and the proposer.

Three Term Recurrence **April 1989**

1318. *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

Define a sequence (p_n) recursively by the formula

$$p_n = \left(\frac{6n-1}{4n}\right)p_{n-1} - \left(\frac{2n-1}{4n}\right)p_{n-2}, \qquad p_0 = 1, \qquad p_1 = 5/4.$$

Evaluate $\lim_{n \to \infty} p_n$.

Solution by Chris Hill, student, Colorado State University, Fort Collins, Colorado.

The desired limit is $\sqrt{2}$.

From the recurrence relation we have

$$p_n - p_{n-1} = \left(\frac{2n-1}{4n}\right)(p_{n-1} - p_{n-2}), \qquad n \geq 2.$$

It quickly follows that

$$p_n - p_{n-1} = \left(\frac{2n-1}{4n}\right)\left(\frac{2n-3}{4(n-1)}\right)\left(\frac{2n-5}{4(n-2)}\right)\cdots\left(\frac{3}{4(2)}\right)\left(\frac{1}{4}\right)$$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{(2n-1)}{2}\right)}{n!}\left(-\frac{1}{2}\right)^n$$

$$= \binom{-\frac{1}{2}}{n}\left(-\frac{1}{2}\right)^n, \qquad n \geq 2.$$

Note that $\binom{-\frac{1}{2}}{n}\left(-\frac{1}{2}\right)^n$ gives us $p_1 - p_0$ when $n = 1$ and it equals p_0 when $n = 0$. Then

$$p_n = (p_n - p_{n-1}) + (p_{n-1} - p_{n-2}) + \cdots + (p_1 - p_0) + p_0$$

$$= \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k}\left(-\frac{1}{2}\right)^k, \qquad n \geq 0.$$

Now, by Newton's binomial theorem

$$(1+x)^{-1/2} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} x^k, \qquad -1 < x < 1.$$

Consequently,

$$\lim_{n\to\infty} \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k}\left(-\frac{1}{2}\right)^k = (1 - 1/2)^{-1/2} = \sqrt{2}.$$

It follows that $\lim_{n\to\infty} p_n$ exists and equals $\sqrt{2}$.

Also solved by Robert A. Agnew, Martin Aub (Jamaica), Seung-Jin Bang (Korea), A. Bhattacharya, Paul Bracken (Canada), W. E. Briggs, David Callan, David G. Cantor, Joseph E. Chance, Chico Problem Group, Con Amore Problem Group (Denmark), Fred Dodd, Robert Doucette, Ervin Eltze, Michael Fichter (West Germany), Kevin Ford (student), Lorraine L. Foster, Guo-Gang Gao (Canada), Jeroen Gijsbers (student), Russell Jay Hendel, J. Heuver (Canada), Hans Kappus (Switzerland), Y. H. Harris Kwong, Lamar University Problem Solving Group, Robert L. Lamphere, Kee-Wai Lau (Hong Kong), Eugene Levine, Stephen C. Locke, David E. Manes, Ricardo Perez Marco (Spain), Reiner Martin (West Germany), Jean-Marie Monier (France), Roger B. Nelsen, William A. Newcomb, Hugh Noland, Stephen Noltie, Michael Parmenter and Bruce Shawyer (Canada), Allan Pedersen (Denmark), Michael J. Poris, J. Metzger and J. Rue, Hyman Rosen, C. O. R. Sarrico and Jorge-Nuno O. Silva (Portugal), Volkhard Schindler (East Germany), Nick Singer, Sahib Singh, Glenn A. Stoops, Jean-Yves Thibon (France), Michael Vowe (Switzerland), J. G. Wendel, Kenneth L. Yocom, Paul J. Zwier, and the proposer. There were four essentially correct solutions that contained minor arithmetical mistakes leading to incorrect answers, and there was one correct conjecture based on computer analysis.

Diophantine Equation, $\sigma(n)=2^m$ **April 1989**

1319. *Proposed by Jeffrey Shallit, Dartmouth College, Hanover, New Hampshire.*

Let $\sigma(N)$ denote the sum of the divisors of N. Show that $\sigma(N)$ is a power of 2 if and only if N is the product of distinct Mersenne primes. (A prime p is Mersenne if $p=2^a-1$ where a is prime.)

I. *Solution by David W. Koster, University of Wisconsin, La Crosse.*

Let $N=p_1^{e_1}p_2^{e_2}\dots p_n^{e_n}$ be the prime power factorization of N, and assume that $\sigma(N)$ is a power of 2. Let p^e denote any of the prime power factors in the above expression for N. Since σ is a multiplicative function, $\sigma(p^e)=1+p+\cdots+p^e$ must be a power of 2. It follows that p and e are odd.

Let $e=2k+1$. Then $\sigma(p^e)=1+p+\cdots+p^e=(1+p)(1+p^2+\cdots+p^{2k})$. Since $p+1$ is a power of 2, we conclude that p is Mersenne.

Suppose that $k>0$. Then from above, $1+p^2+\cdots+p^{2k}$ is a power of 2, and therefore k is odd. Let $k=2m+1$. Then $1+p^2+\cdots+p^{2k}=(1+p^2)(1+p^4+\cdots+p^{4m})$, and thus $1+p^2$ is a power of 2. We conclude that $p^2\equiv-1 \pmod 4$, which is impossible. Thus $k=0$, $e=1$ and N is a product of distinct Mersenne primes.

The converse is immediate, because σ is multiplicative.

II. *Solution by Daniel B. Shapiro, Ohio State University, Columbus, Ohio.*

As above, we need to solve the Diophantine equation

$$1+p+\cdots+p^n=2^m, \qquad (*)$$

where p is prime. If $n=1$ then clearly p is a Mersenne prime. We need to prove that $(*)$ is impossible if $n\geqslant 2$.

One way to see this is to apply the Theorem of Zsigmondy (*Monatshafte Math. Phys.* vol. 3 (1892), pp. 265–284), which says:

Theorem of Zsigmondy. *If a,b,n are whole numbers with a,b coprime and $n\geqslant 2$ then there exists a prime q dividing a^n-b^n but not dividing any a^k-b^k for $k=1,2,\dots,n-1$ except for the one case $a=2$, $b=1$, $n=6$.*

This theorem quickly proves the impossibility of the $(*)$ for $n\geqslant 2$, as follows. Certainly p must be odd, and we have $p^{n+1}-1=2^m(p-1)$ by summing the geometric series. Therefore, any odd prime dividing $p^{n+1}-1$ must already divide $p-1$. This contradicts Zsigmondy's Theorem.

Also solved by Paul Bracken (Canada), Lawrence S. Braden, Seung-Jin Bang (Korea), Kevin Brown, David Callan, Robert Dahlin, Jesse Deutsch, Nicos D. Diamantis (Greece), Fred Dodd, Robert Doucette, Kevin Ford (student), Lorraine L. Foster, S. Gendler, Hans Kappus (Switzerland), H. K. Krishnapriyan, Lamar University Problem Solving Group, Kee-Wai Lau (Hong Kong), David E. Manes, Ricardo Perez Marco (Spain), Helen M. Marston, Reiner Martin (student; West Germany), Jean-Marie Monier (France), Hugh Noland, Daniel E. Otero, Neville Robbins, Hyman Rosen, Heinz-Jürgen Seiffert (West Germany), Shippensburg University Elementary Number Theory Class of 1989, *Jorge-Nuno O. Silva (Portugal), Sahib Singh, Jean-Yves Thibon (France), Edward T. H. Wang (Canada), Jonathan Weinstein (age* 11), *Paul J. Zwier, and the proposer.*

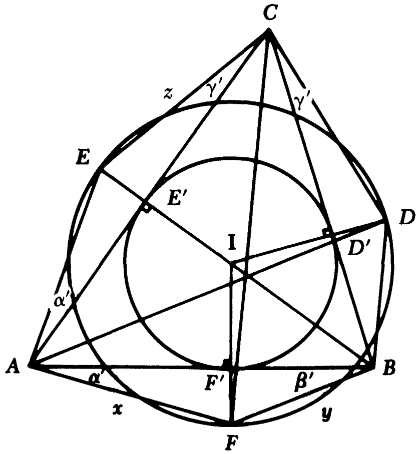
This result is not new. *Shapiro* cites an 1888 paper of Sylvester: "On the divisors of the sum of a geometrical series whose first term is unity and common ratio any positive or negative integer," *Nature* 37 (1888), 417–418; reprinted in *Collected Works of J. J. Sylvester*, v. 4, pp. 625–629. More recently, this, and related results are contained in the following papers: D. Estes, R. Guralnick, M. Schacher and E. Straus, "Equations in prime powers," *Pacific Journal of Mathematics* 118 (1985), 359–367 (see Lemma 1), and G. G. Dandapat, J. L. Hunsucker, and C. Pomerance, "Some new results on odd perfect numbers," *Pacific Journal of Mathematics* 57 (1975), 359–364 (see Theorem A). *Barry Brunson*, Western Kentucky University notes that the problem appears as Problem E2493 in *The American Mathematical Monthly* and the solution (October 1975) refers to the same result in a paper by W. Sierpiński.

Generalization of the Gergonne Point **April 1989**

1320. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

Let $C(I)$ be a circle with center I, the incenter of triangle ABC. Let D, E, F be the points of intersection of $C(I)$ with the lines from I that are perpendicular to sides BC, CA, AB respectively. Show that AD, BE, CF are concurrent.

I. *Solution by J. Heuver, Grande Prairie Composite High School, Grande Prairie, Alberta, Canada.*



Let the intersection of AD, BE, CF with BC, CA, AB be D', E', F', respectively. It is easy to establish that $\angle FAF' = \angle EAE' = \alpha'$, $\angle FBF' = \angle DBD' = \beta'$, $\angle DCD' = \angle ECE' = \gamma'$. Further, $AE = AF = x$, $BF = BD = y$, $CD = CE = z$. The ratio $AF'/F'B$ equals the ratio of the altitudes from A and B on CF of the triangles AFC and BFC and hence as the ratio of their areas. Therefore,

$$\frac{AF'}{F'B} = \frac{\text{Area}\triangle AFC}{\text{Area}\triangle BFC} = \frac{xAC\sin(\angle A + \alpha')}{yBC\sin(\angle B + \beta')}.$$

Similarly,

$$\frac{BD'}{D'C} = \frac{yAB\sin(\angle B + \beta')}{zAC\sin(\angle C + \gamma')}, \qquad \frac{CE'}{E'A} = \frac{zBC\sin(\angle C + \gamma')}{xAB\sin(\angle A + \alpha')}.$$

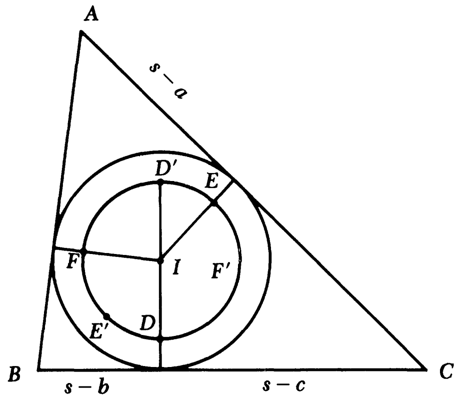
It follows that $\frac{AF'}{F'B} \cdot \frac{BD'}{D'C} \cdot \frac{CE'}{E'A} = 1$, so by Ceva's Theorem, AD, BE, and CF are concurrent.

II. *Solution by Richard E. Pfiefer, San Jose State University, San Jose, California.*

Let $t > 0$ denote the radius of $C(I)$ and $r > 0$ denote the inradius of $\triangle ABC$. We note that there are two points D and D' where the line on I and perpendicular to side BC intersects $C(I)$. Similarly for points E and F (see figure).

The homogeneous trilinear coordinates of a point $P(x, y, z)$ are determined by the signed distances from P to the sides BC, CA, AB respectively, with the distance taken as positive if P is on the same side of BC as A, and similarly for y and z. We can easily compute the trilinear coordinates of D, E, F and D', E', F':

$$\begin{array}{lcl} D(r-t, \gamma, \beta) & & D'(r+t, \gamma', \beta') \\ E(\gamma, r-t, \alpha) & \text{and} & E'(\gamma', r+t, \alpha') \\ F(\beta, \alpha, r-t) & & F'(\beta', \alpha', r+t), \end{array}$$



where

$$\gamma = (s-c)\sin C - (r-t)\cos C$$
$$\gamma' = (s-c)\sin C - (r+t)\cos C$$

and similarly for α, α', and β, β' with $s=(a+b+c)/2$, the semiperimeter of $\triangle ABC$.

Thus the homogeneous line coordinates are

$$\begin{array}{lcl} AD(0,\,-\beta,\gamma) & & AD'(0,\,-\beta',\gamma') \\ BE(\alpha,0,\,-\gamma) & \text{and} & BE'(\alpha',0,\,-\gamma') \\ CF(-\alpha,\beta,0) & & CF'(-\alpha',\beta',0). \end{array}$$

By verifying that the determinant of the 3×3 matrices formed using the rows above is zero, we conclude that AD, BE and CF are concurrent at $P(1/\alpha, 1/\beta, 1/\gamma)$ and AD', BE', and CF' are concurrent at $P'(1/\alpha', 1/\beta', 1/\gamma')$.

Note that $P=P(t)$ is given as a function of the parameter t, the radius of $C(I)$ (and similarly for $P'(t)$). If $\triangle ABC$ is equilateral, $P(t)$ is a single point, but if $\triangle ABC$ is isosceles, then $P(t)$ describes a line. In general, $P(t)$ lies on a conic section which lies on the incenter I, the Gergonne point G and the orthocenter H.

Also solved by Jordi Dou (Spain), Chico Problem Group, Francis M. Henderson, Hans Kappus (Switzerland), Lamar University Problem Solving Group, and the proposer. There were ten incorrect solutions, nine of them caused by taking $C(I)$ to be the incircle.

Dou showed that in the general case, the locus of $P(t)$ (using the notion of *Solution II*) is a rectangular hyperbola on the points A, B, C, and I.

Gamma Function Inequality **April 1989**

1321. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $x, y, z \geqslant \sqrt{3}$. Prove that

$$\Gamma(x+y+z+1) \geqslant (x+1)(y+1)(z+1)\Gamma(x+1)\Gamma(y+1)\Gamma(z+1).$$

Solution by Robert E. Shafer, Berkeley, California.

Without loss of generality, suppose $x \leqslant y \leqslant z$. We introduce the ψ-function of K. Weierstrass,

$$\psi(t) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+t-1}\right) = \frac{d}{dt}\log\Gamma(t).$$

For $x > 1/2$ we have

$$\log \frac{\Gamma(2x+y+1)\Gamma(x+2)}{\Gamma(3x+1)\Gamma(y+2)} = \int_x^y \psi(2x+t+1) - \psi(t+2)\,dt$$
$$= \sum_{n=1}^{\infty} \int_x^y \left(\frac{1}{1+t+n} - \frac{1}{2x+t+n} \right) dt > 0.$$

From this, it follows that

$$\frac{\Gamma(2x+y+1)}{\Gamma^2(x+2)\Gamma(y+2)} > \frac{\Gamma(3x+1)}{\Gamma^3(x+2)}, \qquad x > 1/2. \tag{1}$$

In the same manner, for $x > 1/2$,

$$\log \frac{\Gamma(x+y+z+1)\Gamma(x+2)}{\Gamma(2x+y+1)\Gamma(z+2)} = \int_x^z \psi(x+y+t+1) - \psi(t+2)\,dt$$
$$= \sum_{n=1}^{\infty} \int_x^z \left(\frac{1}{1+t+n} - \frac{1}{x+y+t+n} \right) dt \geqslant 0.$$

It follows that

$$\frac{\Gamma(x+y+z+1)}{\Gamma(x+2)\Gamma(y+2)\Gamma(z+2)} > \frac{\Gamma(2x+y+1)}{\Gamma^2(x+2)\Gamma(y+2)}, \qquad x > 1/2. \tag{2}$$

Combining inequalities (1) and (2), we have

$$\frac{\Gamma(x+y+z+1)}{\Gamma(x+2)\Gamma(y+2)\Gamma(z+2)} \geqslant \frac{\Gamma(3x+1)}{\Gamma^3(x+2)}, \qquad x > 1/2. \tag{3}$$

We have strict inequality if at least one of x, y, z differs from the others.

Now, observe that $\frac{\Gamma(3x+1)}{\Gamma^3(x+2)}$ is an increasing function of x, for $x > 1/2$, because

$$\frac{d}{dx} \log \frac{\Gamma(3x+1)}{\Gamma^3(x+2)} = 3\psi(3x+1) - 3\psi(x+2) = 3 \sum_{n=1}^{\infty} \left(\frac{1}{x+1+n} - \frac{1}{3x+n} \right) > 0.$$

From Abramowitz and Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematical Series, 55 (1964), pp. 268–269, we find that $\Gamma(5/4) = 0.9064024771\ldots$, and $\Gamma(7/4) = 0.9190625268\ldots$. Thus, for $x = 5/4$, we get $\Gamma(3 + 7/4)/\Gamma^3(2 + 5/4) > 1$, which shows that the desired inequality of the problem holds for $x \geqslant 5/4$ and is a strict inequality.

Also solved by Peik Bremer (West Germany), M. S. Klamkin (Canada), Michael Vowe (Switzerland), Paul Bracken (Canada), Irl Bivens, Heinz-Jürgen Seiffert (West Germany), and the proposer. There was one incorrect solution.

Answers

Solutions to the Quickies on p. 126.

A76. Let P be a polynomial with integer coefficients and suppose that b is a *positive* integer zero of P that is larger than every coefficient of P. Transposing the terms of P that are negative (if any) to the right side of the equation $P(b)=0$, we shall have

$$\begin{aligned} &d_n b^n + d_{n-1} b^{n-1} + \cdots + d_1 b + d_0 \\ &\quad = \delta_n b^n + \delta_{n-1} b^{n-1} + \cdots + \delta_1 b + \delta_0, \qquad d_j \delta_j = 0, \end{aligned} \tag{*}$$

where d_j and δ_j are all non-negative integers less than b, $1 \leqslant j \leqslant n$. But (*) contradicts the uniqueness of representation of a number to the base b. We conclude that such a zero cannot exist.

The case of a large *negative* integer zero b can be reduced to the above case by creating a new polynomial $\bar{P}$ such that $\bar{P}(|b|)=0$, where $\bar{P}$ is obtained from P by reversing some signs of coefficients of P as needed.

A761. Letting $P=\prod_{i=1}^{n+1} a_i$, $S=\sum_{i=1}^{n+1} a_i^{-n}$, the inequality can be written as

$$P\left[\left(S-a_1^{-n}\right)+\left(S-a_2^{-n}\right)+\cdots+\left(S-a_{n+1}^{-n}\right)\right] \geqslant n \sum_{i=1}^{n+1} a_i.$$

Since by the arithmetic mean - geometric mean inequality,

$$S-a_i^{-n} \geqslant n a_i/P,$$

we get the desired inequality. There is equality if and only if all the a_i are equal.

Alternative solution with generalization. Letting $a_k=1/x_k$, we then have to show equivalently that

$$x_1^n + x_2^n + \cdots x_{n+1}^n \geqslant x_1 x_2 \cdots x_n + x_2 x_3 \cdots x_{n+1} + \cdots + x_{n+1} x_1 \cdots x_{n-1}$$

for $x_k>0$. More generally, we have

$$x_1^p + x_2^p + \cdots + x_n^p \geqslant x_1 x_2 \cdots x_p + x_2 x_3 \cdots x_{p+1} + \cdots + x_n x_1 \cdots x_{p-1}$$

for all positive integers p and n. A proof follows immediately by applying Hölder's inequality to

$$\left(x_1^p+x_2^p+\cdots+x_n^p\right)^{1/p}\left(x_2^p+x_3^p+\cdots+x_1^p\right)^{1/p}\cdots\left(x_p^p+x_{p+1}^p+\cdots+x_{p+n-1}^p\right)^{1/p}$$

where $x_{n+k}=x_k$.

A762. We construct the number

$$N=a_1 0\ldots 0 a_2 0\ldots 0 a_3 0\ldots 0\ldots a_m 0\ldots 01$$

such that each a_k is in position i_k, and all other positions, except the units position is zero, and the units digit is 1. By Dirichlet's Theorem on primes in arithmetical sequences, we know that the sequence

$$n_k = N + k \times 10^{i_1+1}, \qquad k=1,2,3,\ldots$$

contains an infinite number of primes. This proves the result.

REVIEWS

PAUL J. CAMPBELL, *editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Peterson, Ivars, The color of geometry: Computer graphics adds a vivid new dimension to geometric investigations, *Science News* 136 (23 and 30 December 1989) 408-410, 415.

Scenes and prospects from the Geometry Supercomputer Project, based at the University of Minnesota. Knots, soap bubbles, fractals, a "hyperbolic viewer" (for hyperbolic geometries)—all extend the imagination, encouraging the asking of new questions.

von Baeyer, Hans C., Impossible crystals, *Discover* (February 1990) 69-78, 84.

Quasicrystals are the three-dimensional analogues of the Penrose tilings of the plane, which—though nonperiodic—exhibit symmetry (usually fivefold symmetry). Mathematical quasicrystals were investigated before physical ones were discovered (both in the early 1980s). A key mystery, though, was how millions of real atoms could assemble themselves spontaneously into the nonperiodic patterns. Humans who have played with a set of Penrose pieces have not found it easy to extend a tiling indefinitely; some planning ahead seems to be needed. In fact, though, George Onoda (IBM) and George Steinhardt (U. of Pennsylvania), with two others, fashioned in 1988 a simple algorithm for building a Penrose tiling, without the need for any global strategy or lookahead. Now the task is to generalize the algorithm to three dimensions and show that the steps correspond to rules dictated by atomic forces.

Wilf, Herbert S., *generatingfunctionology*, Academic Pr, 1990; viii + 184 pp.

Based on a course in discrete mathematics for seniors, this book presents generating functions as a bridge between the discrete and the continuous. The author underplays the power of hypergeometric series, exhibits his "Snake Oil" method for evaluating combinatorial sums (and shows it to be mightily effective), and scoops his own research article on proof of combinatorial identities by rational function certification. Every math major and mathematician can learn a lot from this book.

Wells, David, *Hidden Connections, Double Meanings*, Cambridge U Pr, 1988; 164 pp, $14.95 (P).

Collection of mostly geometrical puzzles, emphasizing "insight and imagination rather than technique."

Mangel, Marc, and Colin W. Clark, *Dynamic Modeling in Behavioral Ecology*, Princeton U Pr, 1988; xii + 300 pp, $15.95 (P).

Uses dynamic optimization models (really, stochastic dynamic programming models) to try to explain the behavior of biological organisms. The audience is supposed to be familiar with mathematical probability (reviewed briskly in Chapter 1) but perhaps has never written a computer program (an Addendum gives very elementary guidance on programming in Basic). The first problem investigated is patch selection: a biological organism picks patches in which to forage, based on expected gain (food to be found there), risk (of predation), and the state of fitness of the animal (from well-fed to starving). The authors derive a formula for the corresponding "lifetime fitness function" and an algorithm for dynamically computing it, then urge the reader to code a program for analyzing a prototype model with three states. The results confirm intuitions, and the reader is led through further experiements, with applications to the hunting behavior of lines, reproduction in insects, migrations of aquatic organisms, clutch size in birds, and movement of spiders and raptors.

Rowe, David E., and John McCleary (eds.), *The History of Modern Mathematics.* Vol 1: *Ideas and their Reception.* Vol 2: *Institutions and Applications.* Academic Press, 1989; xvi + 453 pp, $39.50; xvi + 325 pp, $37.50.

Twenty-four essays on 19th- and 20th-century mathematics, from a 1988 symposium at Vassar. Among other topics are Riemann's inaugural address as a philosophical response to Kant, the birth of Lie groups, the role of German mathematical societies in promoting applications of mathematics, and a history of the early approaches to Cantor's continuum problem.

Howson, Colin, and Peter Urbach, *Scientific Reasoning: The Bayesian Approach*, Open Court, 1989; xiii + 312 pp, $16.95 (P).

"[S]cientific reasoning is reasoning in accord with the calculus of probabilities ... . We have written this book ... to convince believers in 'objective' standards in science that there is nothing subjective in the Bayesian theory *as a theory of inference* ... [and] to demonstrate ... that this is the *only* theory which is adequate to the task of placing inductive inference on a sound foundation."

Topsoe, Flemming, *Spontaneous Phenomena: A Mathematical Analysis*, Academic Pr, 1990; xi + 182 pp.

Considers radioactivity and derives the standard Poisson model for it; then compares the model with actual data, finds discrepancies, and modifies the model. Included are exercises, a historical perspective on both the Poisson distribution and on radioactivity, some helpful programs in Basic, and a score of further situations to model. This book is particularly suitable for an interdisciplinary course combining mathematics and physics; it presupposes a calculus background and a willingness to wade into mathematical probability.

Pool, Robert, Chaos theory: how big an advance?, *Science* 245 (7 July 1989), 26-28.

Is chaos "merely an interesting idea enjoying a faddish vogue ... or a revolution in scientific thought?" Will chaos achieve the status of quantum mechanics or relativity as a new scientific paradigm? This article, the last in a six-part series on how scientists are using chaos, lets exponents and detractors of chaos have their say.

NEWS AND LETTERS

50th ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION: WINNERS AND SOLUTIONS

Teams from 288 institutions competed in the 1989 William Lowell Putnam Mathematical Competition. The top five winning teams, in descending rank, are:

Harvard University
Jeremy A. Kahn, Raymond M. Sidney, Eric K. Wepsic

Princeton University
David J. Grabiner, Matthew D. Mullin, Rahul V. Pandharipande

University of Waterloo
Grayden Hazenberg, Stephen M. Smith, Colin M. Springer

Yale University
Bruce E. Kaskel, Andrew H. Kresch, Sihao Wu

Rice University
Hubert L. Bray, John W. McIntosh, David S. Metzler

The six highest ranking individuals, named Putnam Fellows, are

Christos Athanasiadis	Massachusetts Institute of Technology
William P. Cross	California Institute of Technology
Andrew H. Kresch	Yale University
Colin M. Springer	University of Waterloo
Ravi D. Vakil	University of Toronto
Sihao Wu	Yale University

Solutions to the 1989 *Putnam problems were prepared for publication in this* Magazine *by Loren Larson, St. Olaf College.*

A-1. How many primes among the positive integers, written as usual in the base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?

Sol. One. An integer of this type, having n 1's is

$$10^{2n-2} + 10^{2n-4} + \cdots + 10^2 + 1 = \frac{10^{2n} - 1}{10^2 - 1} = \frac{(10^n - 1)(10^n + 1)}{99}.$$

When $n = 2$, the number is prime. When $n > 2$, each factor in the numerator is greater than 99 so the integer is composite.

A-2. Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx$ where a and b are positive.

Sol. Divide the region into two parts by the diagonal line $ay = bx$ to get

$$\int_0^a\int_0^b e^{\max\{b^2x^2,a^2y^2\}}dy\,dx$$

$$= \int_0^a \int_0^{bx/a} e^{b^2x^2} dy\, dx + \int_0^b \int_0^{ay/b} e^{a^2y^2} dx\, dy$$

$$= \int_0^a \frac{bx}{a} e^{b^2x^2}\, dx + \int_0^b \frac{ay}{b} e^{a^2y^2}\, dy = \frac{e^{a^2b^2} - 1}{ab}.$$

A-3. Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

Sol 1. We have $z^9 = \dfrac{11 - 10iz}{11z + 10i}$. If $z = a + bi$, then

$$|z^9| = \left|\frac{11 - 10iz}{11z + 10i}\right|$$

$$= \sqrt{\frac{|11^2 + 220b + 10^2(a^2 + b^2)|}{|11^2(a^2 + b^2) + 220b + 10^2|}}$$

$$\equiv \frac{f(a,b)}{g(a,b)}.$$

If $a^2 + b^2 > 1$, then $g(a, b) > f(a, b)$, making $|z^9| < 1$, a contradiction. If $a^2 + b^2 < 1$, then $f(a, b) > g(a, b)$, making $|z^9| > 1$, a contradiction. Thus, $|z| = 1$.

Sol. 2. Substitute $z = e^{i\theta}$ into the equation, and multiply each side by $e^{-i\theta}$, to get $11(e^{i5\theta} - e^{-i5\theta}) + 10i(e^{i4\theta} + e^{-i4\theta}) = 0$, which in trigonometric form, leads to the equivalent equation $11\sin 5\theta = -10\cos 4\theta$. Set $f(\theta) = 11\sin 5\theta$ and $g(\theta) = -10\cos 4\theta$. f has period $2\pi/5$ and g has period $2\pi/4$, and it is easy to check graphically that the two curves will intersect 10 different times in the interval $[0, 2\pi]$. Thus, the polynomial equation has ten zeros on the unit circle. But the polynomial is of degree ten, so the Fundamental Theorem of Algebra, this is all of the zeros.

A-4. If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both 1/2. A game is finite if with probability 1 it must end in a finite number of moves.)

Sol. Yes. Write the number

$$\alpha = \sum_{n=1}^{\infty} \frac{\alpha_n}{2^n}$$

in its binary representation,

$$\alpha_n = 0 \text{ or } 1, \qquad n = 1, 2, 3, \dots,$$

and play the game as follows. Keep tossing the coin, recording the result, say β_n, of the n-th toss as 0 when it falls tails and 1 when it falls heads; you win the game if the first time that $\beta_n \neq \alpha_n$ your result, β_n, is 0 (and, therefore, $\alpha_n = 1$). In other words, you win if at the first time when the β sequence differs from the α sequence, the β sequence is smaller.

The motivation for this procedure is to think of α as determining the interval $(0, \alpha)$, and think of the coin tosses as determining a number in $[0, 1]$; "win" means "land in the prescribed interval". There are many other solutions; this might be one of the simplest.

How can you win? To see the answer, it helps to introduce some notation: let $n_1, n_2, n_3, \dots$ be the positions of the 1's that occur in the sequence, α. One way to win is to toss so that the β sequence agrees with the α sequence for the first $n_1 - 1$ terms and then disagrees. The probability of that is 1/2 to the power n_1, which is equal to the partial sum up to n_1 of the binary representation of α. The next way to win is to toss so that the β sequence agrees with the α sequence for the first $n_2 - 1$ terms and then disagrees. The probability of that is 1/2 to the n_2, which is equal to the number obtained from the partial sum up to n_2 of the binary representation of α by replacing the first 1 with a 0. These two ways of winning are the beginning of an infinite sequence. The (infinite) sum of the corresponding probabilities is exactly the binary representation of α.

A-5. Let m be a positive integer and let $\mathcal{G}$ be a regular $(2m + 1)$-gon inscribed in the unit circle. Show that there is a positive constant A, independent of m, with the following property. For any point p inside $\mathcal{G}$ there are two distinct vertices v_1 and v_2 of $\mathcal{G}$ such that

$$||p - v_1| - |p - v_2|| < \frac{1}{m} - \frac{A}{m^3}.$$

Here $|s - t|$ denotes the distance between the points s and t.

Sol. The diameter of $\mathcal{G}$ is $2\cos\left(\frac{\pi}{2(2m+1)}\right)$, so all the $|p - v_i|$ for $1 \le i \le 2m + 1$ fall into the interval $\left[0, 2\cos\frac{\pi}{2(2m+1)}\right]$. Hence some two of them differ by at most

$$\frac{2\cos\left(\frac{\pi}{2(2m+1)}\right)}{2m}$$

Let

$$f(m) = \frac{1 - \cos\left(\frac{\pi}{2(2m+1)}\right)}{\left(\frac{1}{m}\right)^2}.$$

This is a positive term sequence with limit $\pi^2/32$ and therefore this sequence has a positive minimum value. Therefore

$$\frac{1 - \cos\left(\frac{\pi}{2(2m+1)}\right)}{\left(\frac{1}{m}\right)^2} > A$$

for some $A > 0$. Hence

$$\cos\left(\frac{\pi}{2(2m+1)}\right) < 1 - \frac{A}{m^2}$$

and the result follows.

A-6. Let $\alpha = 1 + a_1x + a_2x^2 + \dots$ be a formal power series with coefficients in the field of two elements. Let

$$a_n = \begin{cases} 1 & \text{if every block of zeros in the binary expansion of } n \text{ has an even number of zeros in the block,} \\ 0 & \text{otherwise.} \end{cases}$$

(For example, $a_{36} = 1$ because $36 = 100100_2$, and $a_{20} = 0$ because $20 = 10100_2$.) Prove that $\alpha^3 + x\alpha + 1 = 0$.

Sol. We show that $\alpha^4 + x\alpha^2 + \alpha = 0$.

Note that $\alpha^2 = (\sum_{n=0}^{\infty} a_nx^n)^2 = \sum_{n=0}^{\infty} a_nx^{2n}$, $\alpha^4 = \sum_{n=0}^{\infty} a_nx^{4n}$, $x\alpha^2 = \sum_{n=0}^{\infty} a_nx^{2n+1}$.

Therefore

$$\alpha^4 + x\alpha^2 + \alpha = \sum_{\substack{n=0 \\ 2|n, 4\nmid n}}^{\infty} a_nx^n + \sum_{\substack{n=0 \\ 4|n}}^{\infty} (a_{n/4} + a_n)x^n + \sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} (a_n + a_{(n-1)/2})x^n.$$

If $2|n, 4\nmid n$, then $a_n = 0$. If $4|n$ then $a_{n/4} = a_n$, so $a_{n/4} + a_n = 0$. If n is odd, then $a_{(n-1)/2} = a_n$ so $a_n + a_{(n-1)/2} = 0$.

B-1. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a\sqrt{b} + c}{d}$, where a, b, c, d are positive integers.

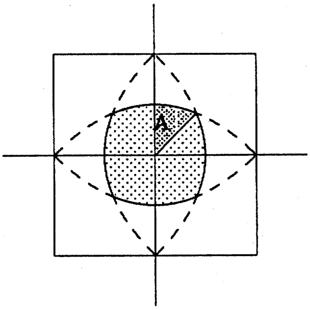
Sol. Consider a 2×2 dartboard centered at the origin with vertices at (1, 1), (-1, 1), (-1, -1), and (1, -1). A point (x, y) in the square is closer to the center of the board than the top edge of the board if and only if

$$\sqrt{x^2 + y^2} \leq 1 - y,$$

or equivalently,

$$y \leq \frac{1 - x^2}{2}.$$

A similar parabolic region arises for each of the other three sides, and therefore the region of points closer to the center than any edge has the shape of the shaded region in the following figure.



Let A denote the area of this region in the first quadrant bounded by $y = (1 - x^2)/2$ and $y = x$. Then, the probability we desire is

$$\frac{8 \text{ Area A}}{4} = 2 \text{ Area A}$$

$$= 2 \int_0^{\sqrt{2}-1} \left(\frac{1 - x^2}{2} - x\right) dx = \frac{4\sqrt{2} - 5}{3}.$$

B-2. Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \dots\}$ is finite. Must S be a group?

Sol. Yes. Let a be an arbitrary element of S. The set $\{a^n : n = 1, 2, 3, \dots\}$ is finite, and therefore $a^m = a^n$ for some m, n with $m > n \geq 1$. By cancellation we have $a^{r(a)} = a$, where $r(a) = m - n + 1 > 1$. If x is any element of S, then $a\, a^{r(a)-1} x = a^{r(a)} x = ax$, and this implies that $a^{r(a)-1} x = x$. Similarly, we see that $x\, a^{r(a)-1} = x$, and the element $e \equiv a^{r(a)-1}$ is an identity. The identity element is unique, for if e' is another identity, then $e = e\, e' = e'$. If $r(a) > 2$ then $a^{r(a)-2}$ is an inverse of a, and if $r(a) = 2$ then $a^2 = a = e$ and a is its own inverse. Thus S is a group.

B-3. Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a non-negative integer, define

$$\mu_n = \int_0^\infty x^n f(x)\, dx$$

(sometimes called the nth moment of f).

a. Express μ_n in terms of μ_0.

b. Prove that the sequence $\left\{\mu_n \frac{3^n}{n!}\right\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

Sol. a. Clearly, $\int_0^\infty x^n g(x)\, dx$ exists for $g(x)$ equal to $f(x)$ or $f'(x)$ because $f(x)$ tends rapidly to 0. Hence

$$\int_0^\infty x^n f'(x) dx = -3\mu_n + 6 \int_0^\infty x^n f(2x) dx.$$

Using parts on the integral on the left and the substitution $u = 2x$ on the integral on the right, we obtain

$$x^n f(x) \Big|_0^\infty - n\mu_{n-1} = -3\mu_n + \frac{6}{2^{n+1}} \mu_n.$$

Since $x^n f(x) \to 0$ as $x \to \infty$ for any $n > 0$ we have

$$\mu_n = \frac{n}{3} \frac{1}{1 - \frac{1}{2^n}} \mu_{n-1}.$$

Iteration now yields

$$\mu_n = \frac{n!}{3^n} \frac{1}{\prod_{m=1}^{n} \left(1 - \frac{1}{2^m}\right)} \mu_0.$$

b. Since $\sum_{m=1}^{\infty} \frac{1}{2^m} < \infty$, the infinite product $\prod_{m=1}^{\infty} \left(1 - \frac{1}{2^m}\right)$ converges to a nonzero finite limit and the result follows.

B-4. Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?

Sol. 1. Yes. For each irrational number x in the interval (0, 1), let $S_x = \{a_1, a_1 + 10a_2, a_1 + 10a_2 + 10^2 a_3, \dots\}$ where $x = .a_1 a_2 a_3 \cdots$ is the decimal expansion of x. Then $|S_x \cap S_y|$ is finite if and only if $x \neq y$.

Sol. 2. Consider the set of all infinite sequences of 0's and 1's. For such a sequence, say $\alpha = (\epsilon_1, \epsilon_2, \epsilon_3, \dots\}$, $\epsilon = 0, 1$, define $S_\alpha = \{P_1, P_2, P_3, \cdots\}$ where $P_n = \prod_{i=1}^n p_i^{\epsilon_i}$, and p_i is the i-th prime ($p_1 = 2$, $p_2 = 3$, $\cdots$). Then $|S_\alpha \cap S_\beta|$ is finite if and only if $\alpha \neq \beta$.

B-5. Label the vertices of a trapezoid T (quadrilateral with two parallel sides) inscribed in the unit circle as A, B, C, D so that AB is parallel to CD and A, B, C, D are in counterclockwise order. Let s_1, s_2 and d denote the lengths of the line segments AB, CD, and OE, where E is the point of intersection of the diagonals of T, and O is the center of the circle. Determine the least upper bound of $\frac{s_1 - s_2}{d}$ over all such T for which $d \neq 0$, and describe all cases, if any, in which it is attained.

Sol. 1. Assume that the bases of the trapezoid are horizontal. We may assume that one base is above the x-axis and the other is on or below. This is because reflecting one of the bases across the horizontal axis does not change the value of $s_1 - s_2$ but the larger trapezoid corresponds to the smaller d value. Suppose that the x-coordinates of the points B and C are real numbers b and c, $0 \leq c < b \leq 1$. Then A has coordinates $(-b, -\sqrt{1 - b^2})$ and C has coordinates $(c, \sqrt{1 - c^2})$. It is clear that $s_1 - s_2$ is $2b - 2c$. The y-intercept of the line through A and C is equal to d. When this is computed and simplified, we find that $(s_1 - s_2)/d = 2(b\sqrt{1 - c^2} + c\sqrt{1 - b^2})$. By setting the two partials equal to zero, and solving, we find that the least upper bound is 2, and it is obtained precisely when $b^2 + c^2 = 1$. Alternatively, if we switch to polar coordinates, and write $b = \cos \phi$ and $c = \cos \theta$, we find that $(s_1 - s_2)/d = 2(\cos \phi \sin \theta + \cos \theta(-\sin \phi)) = \sin(\theta - \phi) \leq 2$, with equality if and only if $\theta - \phi = \pi/2$; that is, if and only if $\angle COB$ is a right angle.

Sol. 2. Assume that the bases of the trapezoids are horizontal, and that the vertices are labeled consecutively, counterclockwise, starting in the third quadrant with A, and let us assume that $AB \geq CD$. Let H be the foot of the perpendicular line from the origin to the diagonal BD, and F the projection of H onto the y-axis. Let $\theta = \angle HOE$. Then $\theta = \angle CDB = \angle CBD$. Let $a = DE$ and $b = EB$. Then $EH = (b - a)/2$, and it follows that $s_1 - s_2 = 2b \cos \theta - 2a \cos \theta = 4((b - a)/2)\cos \theta = 4EH \cos \theta = 4FH$. As θ varies, for a fixed value of d, H lies on a circle with diameter OE. Thus, $s_1 - s_2$ is maximized when $FH = d/2$. It follows that the maximum value of $(s_1 - s_2)/d$ is 2, and it occurs precisely when $\theta = \pi/4$.

B-6. Let $(x_1, x_2, \ldots, x_n)$ be a point chosen at random from the n-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1$. Let f be a continuous function on $[0, 1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i)f(x_{i+1})$$

is $\int_0^1 f(t)P(t)\,dt$, where P is a polynomial of degree n, independent of f, with $0 \leq P(t) \leq 1$ for $0 \leq t \leq 1$.

Sol. The volume in $\mathbf{R}^n$ of all points

$$(x_1, x_2, \ldots, x_n)$$

with $0 < x_1 < x_2 < \cdots < x_n < 1$ is

$$\int_0^1 \int_0^{x_n} \cdots \int_0^{x_2} dx_1\, dx_2 \cdots dx_n = \frac{1}{n!}.$$

Thus, we need to divide

$$M = \sum_{i=0}^{n} \int_0^1 \int_0^{x_n} \cdots \int_0^{x_2} (x_{i+1} - x_i)f(x_{i+1})dx_1 \cdots dx_n$$

by $1/n!$. When $0 \leq i \leq n - 1$ iterated integration yields for a typical summand

$$\int_0^1 \int_0^{x_n} \cdots \int_0^{x_{i+1}} (x_{i+1} - x_i)\frac{x_i^{i-1}}{(i-1)!}f(x_{i+1})dx_i \cdots dx_n$$

$$\int_0^1 \int_0^{x_n} \cdots \int_0^{x_{i+2}} \frac{x_{i+1}^{i+1}}{(i+1)!}f(x_{i+1})dx_{i+1} \cdots dx_n$$

$$\int_0^1 \int_{x_{i+1}}^1 \int_0^{x_n} \cdots \int_{x_{i+1}}^{x_{i+3}} \frac{x_{i+1}^{i+1}}{(i+1)!}f(x_{i+1})dx_{i+2} \cdots dx_n\, dx_{i+1}$$

$$\int_0^1 \frac{(1 - x_{i+1})^{n-(i+1)}}{(n - (i+1))!} \frac{x_{i+1}^{i+1}}{(i+1)!}f(x_{i+1})dx_{i+1}.$$

Multiplication by $n!$ reveals that the kernels are "almost" the terms of the binomial expansion of $((1 - x_{i+1}) + x_{i+1})^n$. In fact by adding in and subtracting out the $i = -1$ term we get

$$n!M = \int_0^1 f(t)(1 - (1 - t)^n)dt + n!J_n$$

where

$$J_n \int_0^1 \int_0^{x_n} \cdots \int_0^{x_2} (1 - x_n)f(1)dx_1 \cdots dx_n$$

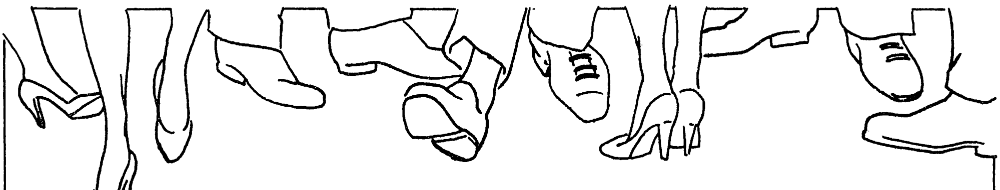
$$= \int_0^1 \int_0^{x_n} (1 - x_n)\frac{x_{n-1}^{n-2}}{(n-2)!} f(1)dx_{n-1}dx_n$$

$$= \frac{1}{(n+1)!} f(1),$$

so the expected value is

$$\int_0^1 f(t)P_n(t)dt + \frac{1}{n+1} f(1)$$

where $P_n(t) = 1 - (1 - t)^n$.



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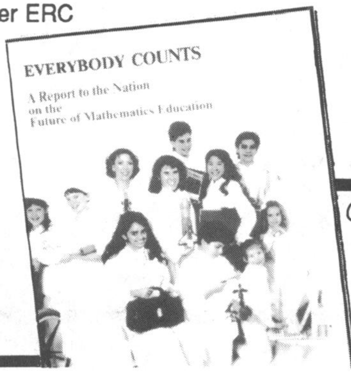

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